International Mathematical Olympiad

Preliminary Selection Contest 2013 — Hong Kong

Outline of Solutions

Answers:

1. 2744	**2.** 45	**3.** $\dfrac{2012}{2013}$	**4.** $\dfrac{-3+\sqrt{22}}{2}$
5. 48	**6.** 7806	**7.** 52	**8.** $2\sqrt{13}$
9. 119	**10.** 3600	**11.** $\dfrac{6039}{8}$	**12.** $\dfrac{3364}{5}$
13. $\dfrac{7\sqrt{3}}{26}$	**14.** $1+\sqrt{15}$	**15.** 96.5°	**16.** $\dfrac{17-4\sqrt{15}}{7}$
17. 33760	**18.** 2	**19.** 10989019	**20.** 7

Solutions:

1. Note that we have $\dfrac{1}{a^3}=\dfrac{8^3}{b^3}=\dfrac{5^3}{c^3}$ and so $a:b:c=1:8:5$. Hence $b=8a$ and $c=5a$. It follows that $d=\dfrac{(a+b+c)^3}{a^3}=\dfrac{(a+8a+5a)^3}{a^3}=14^3=2744$.

2. The common difference may range from 4 to –4. There are 7 numbers with common difference 1 (namely, 123, 234, …, 789), and 5, 3, 1 numbers with common difference 2, 3, 4 respectively. It is then easy to see that there are 8, 6, 4, 2 numbers with common difference –1, –2, –3, –4 respectively (the reverse of those numbers with common difference 1, 2, 3, 4, as well as those ending with 0). Finally there are 9 numbers with common difference 0 (namely, 111, 222, …, 999). Hence the answer is $7+5+3+1+8+6+4+2+9=45$.

3. Computing the first few terms gives $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}=\frac{3}{2}=1+\frac{1}{1\times 2}$, $\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}=\frac{7}{6}=1+\frac{1}{2\times 3}$, $\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}=\frac{13}{12}=1+\frac{1}{3\times 4}$. It is thus reasonable to guess that

$$\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}=1+\frac{1}{k(k+1)}$$

and this indeed is true (and can be verified algebraically). Thus we need only consider the fractional part of

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{2012\times 2013}.$$

Since $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$, the above sum becomes

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{2012}-\frac{1}{2013}\right)=\frac{1}{1}-\frac{1}{2013}=\frac{2012}{2013}$$

which is the answer to the question.

4. Since *x*, *y*, *z* are non-negative we have

$$\frac{13}{4}=x^2+y^2+z^2+x+2y+3z\le(x+y+z)^2+3(x+y+z).$$

Solving the quadratic inequality (subject to *x*, *y*, *z* and hence their sum being non-negative) gives $x+y+z\ge\frac{-3+\sqrt{22}}{2}$. Equality is possible when $x=y=0$ and $z=\frac{-3+\sqrt{22}}{2}$. It follows that the minimum value of $x+y+z$ is $\frac{-3+\sqrt{22}}{2}$.

Remark. Intuitively, since the coefficient of *z* is greater than the coefficients of *x* and *y*, making *z* relatively big and *x*, *y* relatively small could reduce the value of $x+y+z$. As *x*, *y*, *z* are non-negative it would be natural to explore the case $x=y=0$.

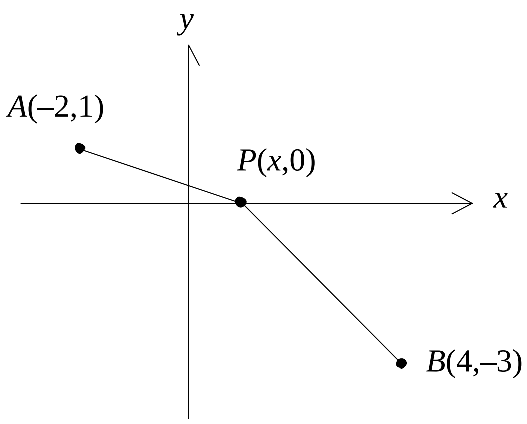
5. Since $22+20+32=74$ and there is exactly one winner each day, we know that the 'certain number' of days in the question is 74. Hence there are $74-22=52$ days on which Peter did not win – he either lost the game (L), or he did not play (N). These must be equal in number (i.e. 26 each), since if we denote each of the 52 days in which Peter did not win by L or N and list them in order, each L must be followed by an N (except possibly the rightmost L) and each N must be preceded by an L (except possibly the first N). Hence the answer is $22+26=48$.

6. As there are no two consecutive 1's, each term in the sum is either equal to 4 (if both multiplicands are 2) or 2 (if the two multiplicands are 1 and 2). Note that the positions of the 1's are 1, 3, 6, 10, …, the triangular numbers. As the 62nd triangular number is $\frac{62\times 63}{2}=1953$ while the 63rd is $1953+63=2016$, exactly 62 of out the first 2014 terms of the sequence are 1 (and the rest are 2).

   In other words exactly 123 terms in the sum are 2 while the rest is 4 (the number 123 comes from $62\times 2-1$, as a_1 only appears in the term a_1a_2, while each other a_i that is equal to 1 appears in two terms, for instance a_3 appears in both a_3a_4 and a_4a_5). It follows that the answer is $4\times 2013-2\times 123=7806$.

7. Let a, b, c, d be among the positive integers. Then both $a+c+d$ and $b+c+d$ are divisible by 39, and so is their difference $a-b$. It follows that any two of the integers are congruent modulo 39. Since $2013=39\times 51+24$, at most 52 integers can be chosen. Indeed, if we choose the 52 integers in the set {13, 52, 91, …, 2002}, then the sum of any three is divisible by 39 (as each one is congruent to 13 modulo 39). It follows that the answer is 52.

8. Rewrite $\sqrt{x^2+4x+5}+\sqrt{x^2-8x+25}$ into the form $\sqrt{(x+2)^2+(0-1)^2}+\sqrt{(x-4)^2+(0+3)^2}$. If we consider the three points $A(-2,1)$, $B(x,0)$ and $P(4,-3)$, then the first term is the distance between A and P while the second is the distance between P and B. The minimum of the sum this occurs when A, P, B are collinear, and the minimum sum is equal to the distance between A and B, which is $\sqrt{(-2-4)^2+[1-(-3)]^2}=2\sqrt{13}$.



9. Rewrite the equation as $10x^3=x^3+3x^2+3x+1=(x+1)^3$. This gives $\frac{x+1}{x}=\sqrt[3]{10}$, or

$$x=\frac{1}{\sqrt[3]{10}-1}=\frac{\sqrt[3]{100}+\sqrt[3]{10}+1}{10-1}=\frac{\sqrt[3]{100}+\sqrt[3]{10}+1}{9}.$$

   It follows that the answer is $100+10+9=119$.

10. The two 0's must not be at the beginning or end. Hence there are $C_2^6=15$ ways to fix the positions of the 0's. It remains to permute the remaining six digits, of which two are the same

(1, 1, 2, 3, 5, 8). There are $\frac{6!}{2!}=360$ such permutations. However, since only 4 of the 6 digits are odd, only two-thirds of these will eventually end up with an odd number. Hence the answer is $15\times 360\times\frac{2}{3}=3600$.

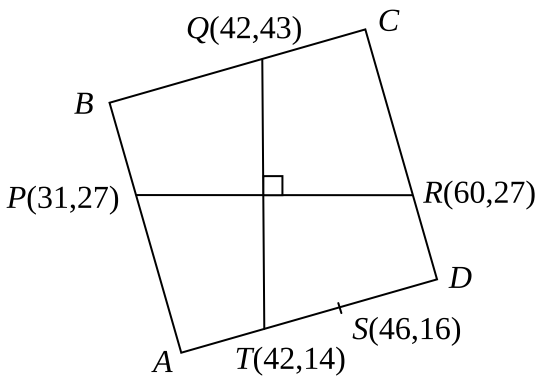
11. Considering the sum of roots gives $\alpha+\beta+\gamma=0$. Hence

$$(\alpha+\beta)^3+(\beta+\gamma)^3+(\gamma+\alpha)^3=(-\gamma)^3+(-\alpha)^3+(-\beta)^3=-\alpha^3-\beta^3-\gamma^3.$$

As α is a root of the equation, we have $8\alpha^3+2012\alpha+2013=0$ and so $-\alpha^3=\frac{2012\alpha+2013}{8}$. Likewise we have $-\beta^3=\frac{2012\beta+2013}{8}$ and $-\gamma^3=\frac{2012\gamma+2013}{8}$ and thus the answer is

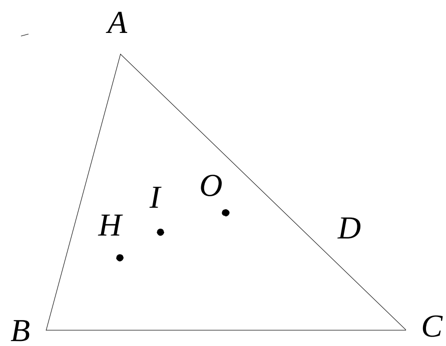
$$\frac{2012\alpha+2013}{8}+\frac{2012\beta+2013}{8}+\frac{2012\gamma+2013}{8}=\frac{2012(\alpha+\beta+\gamma)+6039}{8}=\frac{6039}{8}.$$

12. Let the four given points be *P*, *Q*, *R*, *S* in order. Observe that *PR* is horizontal with length 29. Hence if we draw a vertical line segment from *Q* to meet the square at *T*, the length of *QT* will be 29 as well (think of rotation everything by 90°). Thus *T* has coordinates (42, 14). With coordinates of *S* and *T*, we know that *AD* has slope 0.5 while *BA* and *CD* have slope –2. We can then find that the equations of *AD*, *AB* and *CD* are $x-2y-14=0$, $y=-2x+89$ and $y=-2x+147$ respectively.



Solving the first two equations gives the coordinates of *A* to be (38.4, 12.2), while solving the first and third equations gives the coordinates of *D* to be (61.8, 23.8). It follows that the area of *ABCD* is $AD^2=(61.6-38.4)^2+(23.8-12.2)^2=672.8$.

13. Since $\cos A=\frac{8^2+15^2-13^2}{2(8)(15)}=\frac{1}{2}$, we have $\angle A$ = 60°. As a result we have $\angle BHC=180°-60°=120°$, $\angle BIC=90°+\frac{60°}{2}=120°$ and $\angle BOC=60°\times 2=120°$. Hence *B*, *H*, *I*, *O*, *C* are concyclic. Let *D* be a point on *AC* such that ΔABD is equilateral. Then we have



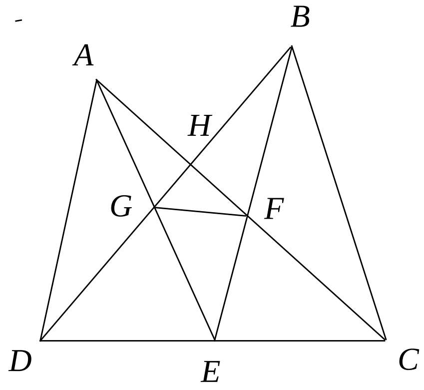
$$\angle HIO=180°-\angle HBO=60°-\angle C=\angle DBC.$$

Applying cosine law in ΔBCD (with BD = 8 and DC = 7), we have

$$\cos\angle DBC = \frac{8^2+13^2-7^2}{2(8)(13)} = \frac{23}{26}$$

and so $\sin\angle HIO = \sin\angle DBC = \sqrt{1-\left(\frac{23}{26}\right)^2} = \frac{7\sqrt{3}}{26}$

14. Let $[PQR]$ denote the area of ΔPQR and $[ABE]=x$. Then $\frac{BF}{FE}=\frac{BG}{GD}=\frac{[ABE]}{[ADE]}=\frac{x}{4}$ and $\frac{AG}{GE}=\frac{AF}{FC}=\frac{[ABE]}{[CBE]}=\frac{x}{3}$.



This yields $[AFE]=\frac{FE}{BE}[ABE]=\frac{4x}{x+4}$ and similarly $[BGE]=\frac{GE}{AE}[ABE]=\frac{3x}{x+3}$.

Using Menelaus' Theorem, we have $\frac{AH}{HF}\cdot\frac{FB}{BE}\cdot\frac{EG}{GA}=1$, i.e. $\frac{AH}{HF}\cdot\frac{x}{x+4}\cdot\frac{3}{x}=1$ or $\frac{AH}{HF}=\frac{x+4}{3}$. This gives

$$[AHE]=\frac{AH}{AF}[AFE]=\frac{4x}{x+7}.$$

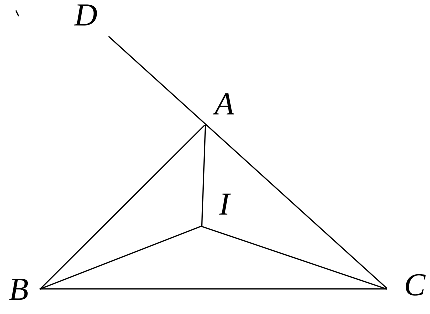
Similarly, $\frac{BH}{HG}\cdot\frac{GA}{AE}\cdot\frac{EF}{FB}=1$ implies $\frac{BH}{HG}\cdot\frac{x}{x+3}\cdot\frac{4}{x}=1$ or $\frac{BH}{HG}=\frac{x+3}{4}$. This gives

$$[BHE]=\frac{BH}{BG}[BGE]=\frac{3x}{x+7}.$$

Using the relation $[ABH]+[AHE]+[BHE]=[ABE]$, we have $2+\frac{4x}{x+7}+\frac{3x}{x+7}=x$. This simplifies to the quadratic equation $x^2-2x-14=0$, which has a unique positive solution $1+\sqrt{15}$.

Remark. The condition that CD is tangent to the circumcircle of ΔABE is not needed. In fact it can be shown that subject to the remaining given conditions such tangency must hold.

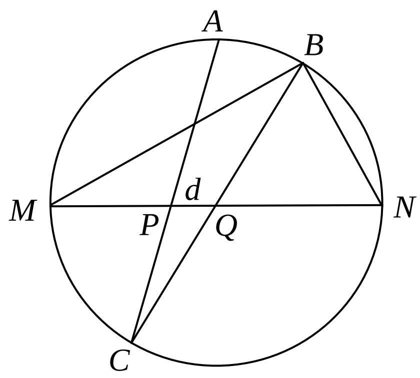
15. Extend CA to D so that $AD=AI$. Join IB, IC and ID. Then we have $BC=AC+AI=AC+AD=CD$. It follows that ΔIBC and ΔIDC are congruent and so



$$\angle BAC = 2\angle IAC = 4ADI = 4\angle IBC = 2\angle ABC.$$

Let $\angle BAC = x$. Then $\angle ABC=\frac{x}{2}$ and $\angle ACB=\frac{x}{2}-13°$. It follows that $x+\frac{x}{2}+\left(\frac{x}{2}-13°\right)=180°$, which gives $x=96.5°$.

16. Let $\frac{CM}{CN}=r$ and d be the length of PQ. Then we have $\frac{MQ}{NQ}=\frac{[BMC]}{[BNC]}=\frac{BM\times CM}{BN\times CN}=\frac{12r}{5}$. Since $MN=1$, we have $NQ=\frac{5}{12r+5}$. Similarly, $\frac{MP}{NP}=\frac{[AMC]}{[ANC]}=\frac{AM\times CM}{AN\times CN}=r$ and this gives $NP=\frac{1}{r+1}$. It follows that



$$d=PQ=\frac{1}{r+1}-\frac{5}{12r+5}.$$

To find the maximum value of *d*, we rewrite the above as a quadratic equation in *r*, namely, $12dr^2+(17d\ ?\quad)r+5d=0$. Since *r* is a real number, the discriminant must be non-negative, i.e. $(17d-7)^2-4(12d)(5d)\geq 0$. Bearing in mind that $d<1$, solving the inequality gives $d\leq\frac{17-4\sqrt{15}}{7}$ and it is easy to check that such maximum is indeed attainable (by choosing the corresponding value of *r* which determines the position of *C*).

17. Since $50688=2^9\times 99$, we must have $m+n=2^k\times 99$ where *k* is one of 0, 2, 4, 6, 8. Forgetting about $m\neq n$ for the moment, there are $2^k\times 99+1$ choices of *m* for each *k*, as *m* can range from 0 to $2^k\times 99$. This leads to a total of $(2^0+2^2+2^4+2^6+2^8)\times 99+5=33764$ pairs of (*m*, *n*). Among these, 4 pairs violate the condition $m\neq n$, as $m=n$ is possible only when *k* is 2, 4, 6 or 8. Hence the answer is $33764-4=33760$.

18. Let p_n be the number of *n*-digit 'good' positive integers ending with 1 and q_n be the number of *n*-digit 'good' positive integers ending with 2. Then $a_n=p_n+q_n$. Furthermore a 'good' positive integer must end with 21, 211, 2111, 12 or 122. This leads to the recurrence relations $p_n=q_{n-1}+q_{n-2}+q_{n-3}$ as well as $q_n=p_{n-1}+p_{n-2}$. It follows that

$$p_n=(q_{n-2}+q_{n-3})+(q_{n-3}+q_{n-4})+(q_{n-4}+q_{n-5})=q_{n-2}+2q_{n-3}+2q_{n-4}+q_{n-5}$$
$$q_n=(p_{n-2}+p_{n-3}+p_{n-4})+(p_{n-3}+p_{n-4}+p_{n-5})=p_{n-2}+2p_{n-3}+2p_{n-4}+p_{n-5}$$

Adding gives $a_n=a_{n-2}+2a_{n-3}+2a_{n-4}+a_{n-5}$. In particular $a_{10}=a_8+a_5+2(a_7+a_6)$ and so

$$\frac{a_{10}-a_8-a_5}{a_7+a_6}=2.$$

19. We have $0.123456789 \le \frac{p}{q} < 0.12345679$. Multiplying both sides by 81 gives $9.999999909 \le \frac{81p}{q} < 9.99999999$, or equivalently, $0.000000091 \ge \frac{10q-81p}{q} > 0.00000001$. Since $10q-81p$ is an integer, it is at least 1 and so $q \ge \frac{1}{0.000000091} > 10989010$.

When $10q-81p=1$, then $10q-1$ is divisible by 81 and is at least 109890099. The smallest multiple of 81 above this minimum is 109890189. This corresponds to $q=10989019$ and $p=1356669$. This q is clearly smallest when $10q-81p=1$, and is also smallest in general since if $10q-81p>1$ we would have $q \ge \frac{10q-81p}{0.000000091} \ge \frac{2}{0.000000091} > 20000000$. It follows that the smallest possible value of q is 10989019.

Remarks.

(1) It can be checked that $\frac{1356669}{10989019} = 0.1234567890000008\ldots$.

(2) The upper bound $\frac{p}{q} < 0.12345679$ is basically not used in the solution. However if one recalls that $111111111 = 12345679 \times 9$, then this would lead us to consider multiplication by 81. It also helps compute 0.123456789×81 more easily.

20. Rewrite the given equation as $(a+b)^2 + 16(a-b)^2 = 16$. Hence we may let $a+b=4\cos x$ and $a-b=\sin x$. Note that

$$\sqrt{16a^2+4b^2-16ab-12a+6b+9} = \sqrt{(4a-2b)^2-3(4a-2b)+9} = \sqrt{\left(4a-2b-\frac{3}{2}\right)^2+\frac{27}{4}}.$$

Since $4a-2b=(a+b)+3(a-b)=4\cos x+3\sin x$, whose value lies between –5 and 5, the maximum value of the above expression occurs when $4a-2b=-5$, and the maximum value is $\sqrt{\left(-5-\frac{3}{2}\right)^2+\frac{27}{4}}=7$. (The corresponding values of a and b can be found by solving the equations $4a-2b=-5$ and $(a+b)^2+16(a-b)^2=16$, giving $a=-\frac{19}{10}$ and $b=-\frac{13}{10}$.)