香港資優教育學苑
The Hong Kong Academy for Gifted Education

International Mathematical Olympiad Preliminary Selection Contest - Hong Kong 2013

2013 國際數學奧林匹克 – 香港選拔賽

18 May 2013 (Saturday)
2013 年 5 月 18 日（星期六）

Question Book
問題簿

Instructions to Contestants:
考生須知：

1. The contest comprises a 3 hours written test.
   比賽以筆試形式進行，限時三小時。

2. Questions are in bilingual versions. Contestants should answer all questions.
   題目中英對照。參賽學生必須解答全卷所有題目。

3. Put your answers on the answer sheet.
   請將答案寫在答題紙上。

4. The use of calculators is NOT allowed.
   不可使用計算機。

5. Measuring instruments like rulers, compasses, etc. can be used.
   直尺、圓規及其它量度工具可作輔助之用。

Co-organised by The Hong Kong Academy for Gifted Education,
the Gifted Education Section of the Education Bureau and
International Mathematical Olympiad Hong Kong Committee
香港資優教育學苑、教育局資優教育組及國際數學奧林匹克香港委員會合辦

1. Let *a*, *b*, *c*, *d* be positive numbers such that $\frac{1}{a^3}=\frac{512}{b^3}=\frac{125}{c^3}=\frac{d}{(a+b+c)^3}$. Find *d*. (1 mark)

設 a、b、c、d 為正數，使得 $\frac{1}{a^3}=\frac{512}{b^3}=\frac{125}{c^3}=\frac{d}{(a+b+c)^3}$ 。求 d。（1 分）

2. How many three-digit positive integers are there such that, the three digits of every integer, taken from left to right, form an arithmetic sequence? (1 mark)

有多少個三位正整數的三個數字從左至右看時組成一個等差數列？（1 分）

3. Let $x=\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{2012^2}+\frac{1}{2013^2}}$. Find the value of $x-[x]$, where $[x]$ denotes the greatest integer not exceeding *x*. (1 mark)

設 $x=\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{2012^2}+\frac{1}{2013^2}}$ 。求 $x-[x]$ 的值（這裡 $[x]$ 表示不超過 x 的最大整數）。（1 分）

4. Let *x*, *y*, *z* be non-negative numbers such that $x^2+y^2+z^2+x+2y+3z=\frac{13}{4}$. Find the minimum value of $x+y+z$. (1 mark)

設 x、y、z 為非負數，使得 $x^2+y^2+z^2+x+2y+3z=\frac{13}{4}$。求 $x+y+z$ 的最小值。（1 分）

5. Peter, Paul and David joined a table tennis tournament. On the first day, two of them were randomly chosen to play a game against each other. On each subsequent day, the loser of the game on the previous day would take a rest and the other two persons would play a game against each other. After a certain number of days, it was found that Peter had won 22 games, Paul had won 20 games and David had won 32 games. What was the total number of games that Peter had played? (1 mark)

小明、小強和小輝參加了一個乒乓球比賽。在第一天，在他們之間隨意抽出兩人比賽一場；在之後的每一天，均是之前一天比賽的負方休戰，而另外兩人則比賽一場。若干天後，發現小明共勝出 22 次、小強共勝出 20 次、小輝共勝出 32 次。那麼，小明共比賽了多少場？（1 分）

6. The sequence 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, … is formed as follows: write down infinitely many ‘1’s, insert a ‘2’ between the first and the second ‘1’s, insert two ‘2’s between the second and the third ‘1’s, insert three ‘2’s between the third and the fourth ‘1’s, and so on. If a_n denotes the *n*-th term of the sequence, find the value of $a_1a_2+a_2a_3+\cdots+a_{2013}a_{2014}$. (1 mark)

數列 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, … 的組成方法如下：先寫下無限個「1」，然後在第一個和第二個「1」之間插入一個「2」，再在第二個和第三個「1」之間插入兩個「2」，繼而在第三個和第四個「1」之間插入三個「2」，如此類推。若 a_n 表示數列的第 n 項，求 $a_1a_2+a_2a_3+\cdots+a_{2013}a_{2014}$ 的值。（1 分）

7. There are n different positive integers, each one not greater than 2013, with the property that the sum of any three of them is divisible by 39. Find the greatest value of n. (1 mark)

現有 n 個互不相同且每一個都不大於 2013 的正整數，且當中任意三個之和均可被 39 整除。求 n 的最大值。（1 分）

8. If x is a real number, find the smallest value of $\sqrt{x^2+4x+5}+\sqrt{x^2-8x+25}$. (1 mark)

若 x 是實數，求 $\sqrt{x^2+4x+5}+\sqrt{x^2-8x+25}$ 的最小值。（1 分）

9. The equation $9x^3-3x^2-3x-1=0$ has a real root of the form $\frac{\sqrt[3]{a}+\sqrt[3]{b}+1}{c}$, where a, b, c are positive integers. Find the value of $a+b+c$. (1 mark)

方程 $9x^3-3x^2-3x-1=0$ 有一個形如 $\frac{\sqrt[3]{a}+\sqrt[3]{b}+1}{c}$（其中 a、b、c 為正整數）的實根。求 $a+b+c$ 的值。（1 分）

10. By permuting the digits of 20130518, how many different eight-digit positive odd numbers can be formed? (1 mark)

若把 20130518 的數字重新排列，可以得到多少個不同的八位正奇數？（1 分）

11. Let α, β and γ be the three roots of the equation $8x^3+2012x+2013=0$. Find the value of $(\alpha+\beta)^3+(\beta+\gamma)^3+(\gamma+\alpha)^3$. (2 marks)

設 α、β 和 γ 為方程 $8x^3+2012x+2013=0$ 的三個根。求 $(\alpha+\beta)^3+(\beta+\gamma)^3+(\gamma+\alpha)^3$ 的值。（2 分）

12. $ABCD$ is a square on the rectangular coordinate plane, and (31, 27), (42, 43), (60, 27) and (46, 16) are points on its sides AB, BC, CD and DA respectively. Find the area of $ABCD$. (2 marks)

$ABCD$ 是直角座標平面上的一個正方形，而 (31, 27)、(42, 43)、(60, 27) 和 (46, 16) 分別是邊 AB、BC、CD 和 DA 上的點。求 $ABCD$ 的面積。（2 分）

13. In ΔABC, $AB = 8$, $BC = 13$ and $CA = 15$. Let H, I, O be the orthocentre, incentre and circumcentre of ΔABC respectively. Find $\sin\angle HIO$. (2 marks)

在 ΔABC 中，$AB = 8$、$BC = 13$、$CA = 15$。設 H、I、O 分別為 ΔABC 的垂心、內心和外心。求 $\sin\angle HIO$。（2 分）

14. Let $ABCD$ be a convex quadrilateral and E be a point on CD such that the circumcircle of ΔABE is tangent to CD. Suppose AC meets BE at F, BD meets AE at G, and AC meets BD at H. If $FG \parallel CD$, and the areas of ΔABH, ΔBCE and ΔADE are 2, 3 and 4 respectively, find the area of ΔABE. (2 marks)

設 $ABCD$ 為凸四邊形，E 為 CD 上的一點，使得 ΔABE 的外接圓與 CD 相切。設 AC 與 BE 交於 F；BD 與 AE 交於 G；AC 與 BD 交於 H。若 $FG \parallel CD$，且 ΔABH、ΔBCE 和 ΔADE 的面積分別是 2、3、4，求 ΔABE 的面積。（2 分）

15. Let I be the in-centre of ΔABC. If $BC = AC + AI$ and $\angle ABC - \angle ACB = 13°$, find $\angle BAC$. (2 marks)

設 I 為 ΔABC 的內心。若 $BC = AC + AI$ ，且 $\angle ABC - \angle ACB = 13°$ ，求 $\angle BAC$ 。（2 分）

16. A, B, C, M, N are points on the circumference of a circle with MN as a diameter. A, B are on the same side of MN and C is on the other side, with A being the mid-point of arc MN. CA and CB meet MN at P and Q respectively. If $MN = 1$ and $MB = \frac{12}{13}$, find the greatest length of PQ. (2 marks)

A、B、C、M、N 為某個圓的圓周上的點，而 MN 為該圓的直徑。A、B 位於 MN 的同一方，而 C 則位於 MN 的另一方，且 A 為弧 MN 的中點。CA 和 CB 分別與 MN 交於 P 和 Q。若 $MN = 1$ 而 $MB = \frac{12}{13}$，求 PQ 的長度的最大值。（2 分）

17. How many pairs (m, n) of non-negative integers are there such that $m \neq n$ and $\frac{50688}{m+n}$ is an odd positive power of 2? (2 marks)

有多少對非負整數 (m, n) 滿足 $m \neq n$ 且 $\frac{50688}{m+n}$ 是 2 的正奇數冪？（2 分）

18. A positive integer is said to be ‘good’ if each digit is 1 or 2 and there is neither four consecutive 1’s nor three consecutive 2’s. Let a_n denote the number of n-digit positive integers that are ‘good’. Find the value of $\frac{a_{10} - a_8 - a_5}{a_7 + a_6}$. (2 marks)

若某正整數的每個數字均為 1 或 2，且當中既沒有四個連續的「1」亦沒有三個連續的「2」，便稱為「好數」。設 a_n 表示 n 位「好數」的數目。求 $\frac{a_{10} - a_8 - a_5}{a_7 + a_6}$ 的值。（2 分）

19. Let p and q be positive integers. If $\frac{p}{q} = 0.123456789\ldots$ (i.e. when $\frac{p}{q}$ is expressed as a decimal the first 9 digits after the decimal point are 1 to 9 in order), find the smallest value of q. (2 marks)

設 p 、 q 為正整數。若 $\frac{p}{q} = 0.123456789\ldots$（即 $\frac{p}{q}$ 以小數表示時小數點後首 9 個位依次是 1 至 9），求 q 的最小值。（2 分）

20. Let a and b be real numbers such that $17(a^2 + b^2) - 30ab - 16 = 0$. Find the maximum value of $\sqrt{16a^2 + 4b^2 - 16ab - 12a + 6b + 9}$. (2 marks)

設 a 、 b 為實數，使得 $17(a^2 + b^2) - 30ab - 16 = 0$ 。求 $\sqrt{16a^2 + 4b^2 - 16ab - 12a + 6b + 9}$ 的最大值。（2 分）

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