International Mathematical Olympiad

Preliminary Selection Contest 2018 — Hong Kong

Outline of Solutions

Answers:

1.	12	**2.**	167334	**3.**	30°	**4.**	$36-20\sqrt{3}$
5.	194	**6.**	163	**7.**	207360	**8.**	220
9.	1353	**10.**	510050	**11.**	2991	**12.**	2039190
13.	$14-6\sqrt{2}$	**14.**	193	**15.**	400	**16.**	$\frac{49}{16}$
17.	$\frac{18}{19}$	**18.**	$\frac{43}{128}$	**19.**	264	**20.**	10

Solutions:

1. Each of 17, 19 and 23 can only pair up with 1 to form a fraction that can be simplified to an integer. As the integer 1 can only be used once, at least one fraction cannot be simplified to an integer. Therefore, at most 12 fractions can be simplified to integers. This upper bound is attainable, e.g. we may form the fractions

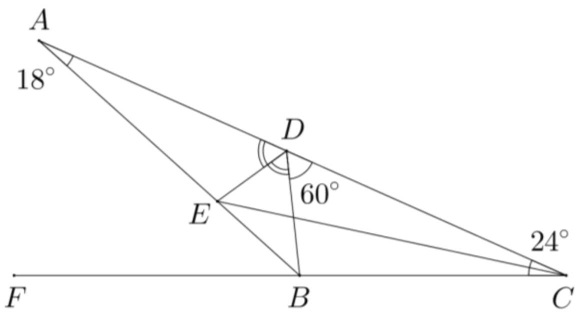
$$\frac{23}{1}, \frac{14}{2}, \frac{15}{3}, \frac{12}{4}, \frac{25}{5}, \frac{24}{6}, \frac{21}{7}, \frac{16}{8}, \frac{18}{9}, \frac{20}{10}, \frac{22}{11}, \frac{26}{13}, \frac{19}{17}$$

in which all but the last can be simplified to an integer.

2. Let the two three-digit numbers be m and n respectively. Then the six-digit number is $1000m+n$. It is given that $1000m+n=3mn$, which is the same as $(3n-1000)m=n$.

Let $3n-1000=k$. Then $1000=3n-k=3km-k=k(3m-1)$. As $3m-1\geq 3(100)-1=299$ and $3m-1$ is a factor of 1000, we must have $3m-1=500$ or $3m-1=1000$. The latter does not have integer solution, while the former gives $m=167$. In that case, we have $k=2$ and hence $n=334$. The six-digit number is thus 167334.

3. Let F be a point on the extension of CB beyond B. From the given angles, we find that $\angle DBA = \angle BDC - \angle BAC = 60° - 18° = 42°$ and $\angle ABF = \angle BAC + \angle ACB = 18° + 24° = 42°$.



These show BE is the angle bisector of $\angle DBF$. As DE is the angle bisector of $\angle ADB$, point E is the ex-centre of ΔCDB opposite to C.

Therefore, we have $\angle ACE = \frac{1}{2}\angle ACB = 12°$. Thus, $\angle BEC = \angle EAC + \angle ACE = 18° + 12° = 30°$.

Remark. This question may also be solved without knowledge of the ex-centre. If we let D' denote the image of the reflection of D across AB, then similar computations as above show that D' lies on the extension of CB and that E is the in-centre of $\Delta AD'C$.

4. From the condition, we find that $f(x+4) = -f(x+2) = f(x)$. This shows f has period 4. As $10\sqrt{3} \approx 17.3$, we have $f(10\sqrt{3}) = f(10\sqrt{3} - 4\times 4) = f\left((10\sqrt{3} - 18) + 2\right) = f(18 - 10\sqrt{3})$. Note that $0 < 18 - 10\sqrt{3} < 1$. Thus, we obtain $f(10\sqrt{3}) = 2(18 - 10\sqrt{3}) = 36 - 20\sqrt{3}$.

5. Let a be the L.C.M. of 1, 2, 3, …, n, and let b be the L.C.M. of 101, 102, 103, …, n. Firstly, observe that the prime number 97 is a factor of a since $n > 100$. If we want to have $a = b$, then 97 must be a factor of one of 101, 102, 103, …, n. Since $97 \times 2 = 194$, we need $n \geq 194$.

We now show that $a = b$ when $n = 194$. It suffices to show that that each of 1, 2, 3, …, 100 divides b. This certainly is true for each of 1, 2, 3, …, 97, since each of these numbers has at least one multiple among 101, 102, … 194. Now $98 = 2\times 7^2$, $99 = 3^2 \times 11$ and $100 = 2^2 \times 5^2$, so it remains to check that $2^2 \times 3^2 \times 5^2 \times 7^2 \times 11$ divides b. This is certainly true as there are multiples of 2^2, 3^2, 5^2, 7^2 and 11 among 101, 102, … 194. Hence the answer is 194.

6. Clearly, $p > 3$. Hence p is odd and it is not divisible by 3, so both $p-1$ and $p+1$ are even and exactly one of them is a multiple of 3. Recall that for any integer n with prime factorisation $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$, it has exactly $(a_1+1)(a_2+1)\cdots(a_t+1)$ positive factors. We have two cases.

- If $p+1$ is a multiple of 3, then $p+1$ must be $2^2 \times 3$ or 2×3^2 since has exactly 6 positive factors. These give $p-1$ to be 10 or 16, neither of which has exactly 10 positive factors.
- If $p-1$ is a multiple of 3, then $p-1$ must be $2^4 \times 3$ or 2×3^4 as it has exactly 10 positive factors. The former case gives $p = 49$ which is not prime. The latter case gives $p = 163$ which is prime, and in that case $p+1 = 164 = 2^2 \times 41$ has exactly 6 positive factors.

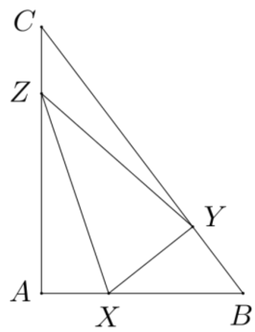
It follows that we have $p = 163$.

7. There are 9 ways to put the number 5. After the position of 5 is fixed, there are 8 ways to choose the number next to 5 in the same row (if 5 is in the middle of a row, then we choose the number on the left of 5). As 5 is the median of its row, there are 4 ways to choose the remaining number of the row (e.g. if 4 is chosen next to 5, then the remaining number in the same row must be chosen from 6 to 9). This gives us $9\times 8\times 4 = 288$ ways of completing the row containing the number 5.

   Now no matter how we fill in the 6 remaining cells, the median of the circled numbers must be 5. (If the median of the circled numbers is greater than 5, that means the medians of the other two rows are both greater than 5, which means there should be two of 6, 7, 8, 9 in each row, contradicting the fact that one of them is in the row containing 5. Similar contradiction arises if the median of the circled numbers is less than 5.) Hence the answer is $288\times 6! = 207360$.

8. Let $P(x) = ax^3 + bx^2 + cx + d$. The problem is equivalent to counting integer solutions to $-a+b-c+d=-9$ subject to $0\le a,b,c,d\le 9$. Setting $x=9-a$ and $y=9-c$, this is in turn equivalent to counting integer solutions to $b+d+x+y=9$ subject to $0\le b,d,x,y\le 9$, i.e. we have to count the number of non-negative integer solutions to $b+d+x+y=9$. It is well-known that the number of such solutions is $H_9^4 = C_{4-1}^{9+4-1} = C_3^{12} = 220$.

9. Let $[DEF]$ denote the area of ΔDEF. We first note that $\angle BAC = 90^\circ$ since $9^2+12^2=15^2$. Thus $[ABC] = \frac{9\times 12}{2} = 54$. Let $AX = BY = CZ = x$. Then we have $AZ = 12-x$, $BX = 9-x$ and $CY = 15-x$. Hence we have $[AXZ] = \frac{AX\times AZ}{2} = \frac{x(12-x)}{2}$ and



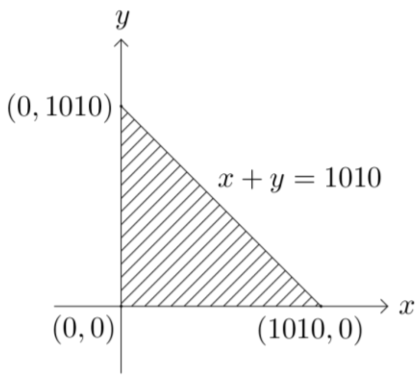
$$[BYX] = \frac{1}{2}\times BY\times BX\sin\angle CBA = \frac{x(9-x)}{2}\times\frac{12}{15} = \frac{2x(9-x)}{5}.$$

   Similarly, $[CZY] = \frac{1}{2}\times CZ\times CY\sin\angle ACB = \frac{x(15-x)}{2}\times\frac{9}{15} = \frac{3x(15-x)}{10}$. It follows that

$$[XYZ] = [ABC]-[AXZ]-[BYX]-[CZY] = 54-\frac{x(141-12x)}{10} = \frac{1}{10}(12x^2-141x+540)$$

   with $0<x<9$. The minimum of this quadratic polynomial occurs when $x=\frac{141}{12\times 2}\approx 5.9$, in which case $[XYZ] = \frac{1}{10}\left[12\times\frac{141^2}{4}-\frac{141^2}{24}+540\right] = \frac{2013}{160}\approx 12.6$. On the other hand, when x approaches 0, ΔXYZ can be arbitrarily close to ΔABC. Thus, $[XYZ]$ can be arbitrarily close to 54 but cannot exceed or be equal to 54. Hence it can take any integer value from 13 to 53 inclusive. The answer is thus $13+14+\cdots+53=1353$.

10. If $x+y<1010$, then $[x]+[y]\le x+y<1010$ and hence $[x]+[y]\le 1009$. In this case, we must have $x+y+[x]+[y]<1010+1009=2019$. This means all points (x,y) where $x,y\ge 0$ and $x+y<1010$ are coloured blue.



If $x+y>1010$, then $[x]+[y]>x-1+y-1>1008$ and hence $[x]+[y]\ge 1009$. In this case, we must have $x+y+[x]+[y]>1010+1009=2019$. This means all points (x,y) where $x,y\ge 0$ and $x+y>1010$ are not coloured blue.

Points (x,y) with $x+y=1010$ do not matter as they lie on a straight line which contributes zero area. Thus, the blue region is an isosceles right-angled triangle with legs 1010. Its area is

$$\frac{1010\times 1010}{2}=510050.$$

11. We have $2^{27653}-1=2^3\times(2^{10})^{2765}-1=8(1025-1)^{2765}-1=8(-1+C_1^{2765}\times 1025+1025^2 m)-1$ for some integer m in view of the binomial theorem. Note that m must be even since $(2^{10})^{2765}$ is even. Let $m=2k$. Then

$$2^{27653}-1=8(-1+2765\times 1025+1025^2\times 2k)-1=22672991+2^4\times 5^4\times 41^2 k.$$

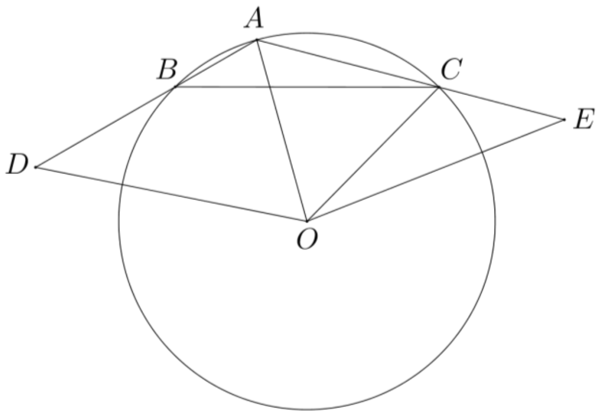
As $2^4\times 5^4$ is a multiple of 10000, the last four digits of $2^{27653}-1$ are 2991.

12. A term in the expansion of $(a^2+20ab+18)^{2018}$ is of the form $(a^2)^i(20ab)^j(18)^{2018-i-j}$. If this and another term of the form $(a^2)^m(20ab)^m(18)^{2018-m-n}$ are like terms, we can compare the powers of a and b to get $2i+j=2m+n$ and $j=n$ respectively. This is possible only if $(i,j)=(m,n)$. This shows distinct choices of the exponents i and j give distinct monomials.

Hence the answer would be the same if we consider the number of unlike terms in the expansion of $(x+y+z)^{2018}$. As each term is of the form $x^i y^j z^{2018-i-j}$, it is the same as counting the number of non-negative integer pairs (i,j) such that $i+j\le 2018$. For each fixed i, there are $2019-i$ choices of j. Hence, the answer is $2019+2018+\cdots+1=2039190$.

Remark. We may also write a general term in the expansion of $(x+y+z)^{2018}$ as $x^i y^j z^k$ where $i+j+k=2018$. Then we would be counting the number of non-negative integer solutions to the equation $i+j+k=2018$, which is $H_{2018}^3=C_{2018}^{2020}=C_2^{2020}=2039190$.

13. Let $DB = x$ and $EC = \frac{x}{\sqrt{2}}$. Since D and E are equidistant from O, their powers with respect to the circumcircle of ΔABC are equal. This implies $DB \times DA = EC \times EA$, so we have $x(x+1) = \frac{1}{\sqrt{2}} x \left(\frac{1}{\sqrt{2}} x + 2 \right)$, solving which gives $x = 2\sqrt{2} - 2$ as $x \neq 0$. Thus, the power of D with respect to the circumcircle of ΔABC is $DB \times DA = x(x+1) = 10 - 6\sqrt{2}$. Recall that the power is also equal to $OD^2 - OA^2$. Thus we have $OD^2 = 10 - 6\sqrt{2} + OA^2 = 14 - 6\sqrt{2}$.



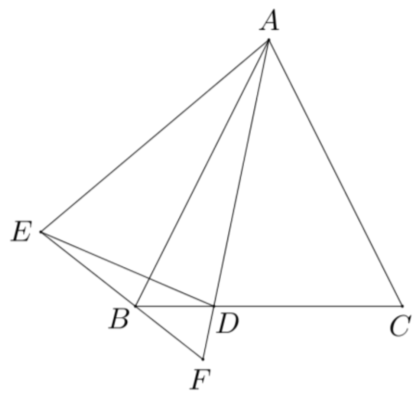
14. Note that 2016 divides $(3n^3 - 2019) + 2016 = 3(n^3 - 1)$. This implies $672 = 2^5 \times 3 \times 7$ divides $n^3 - 1 = (n-1)(n^2 + n + 1)$.

As $n^2 + n + 1$ is odd (because $n^2 + n = n(n+1)$ is the product of two consecutive positive integers, one of which must be even), $n - 1$ must be a multiple of 2^5. Furthermore, $n - 1$ must be a multiple of 3 as well, since if $n \equiv 0$ or $n \equiv 2$ (mod 3), we would have $n^2 + n + 1 \equiv 1$ (mod 3). This means $n - 1$ is divisible by $2^5 \times 3 = 96$.

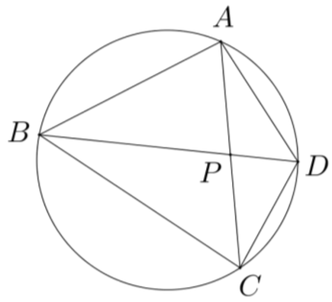
Clearly $n \neq 1$ as $3n^3 - 2019$ is positive. Next we try $n = 97$, which fails as neither $n - 1$ nor $n^2 + n + 1$ is divisible by 7 in this case. The next candidate is $n = 96 \times 2 + 1 = 193$, which works as $n^2 + n + 1 \equiv 2^2 + 2 + 1 \equiv 0$ (mod 7). It follows that the answer is 193.

15. Since $AB = AC$, we have $\angle ABD = \angle ACD = \angle AED$. This implies A, E, B and D are concyclic. Hence, we have $\angle FBD = \angle EAD = \angle CAD$, which in turn shows that A, B, F and C are concyclic. It follows that $\angle ABC = \angle ACB = \angle AFB$, so we have $\Delta ABD \sim \Delta AFB$ and hence $AD \times AF = AB^2 = 400$.

Remark. The condition $BC = 18$ is redundant.



16. Denote by $[XYZ]$ the area of ΔXYZ. Let P be the intersection of AC and BD. Then $\frac{AP}{CP}=\frac{[ABD]}{[CBD]}=\frac{\frac{1}{2}AB\times AD\sin\angle BAD}{\frac{1}{2}CB\times CD\sin\angle BCD}=\frac{\sin\angle BAD}{\sin\angle BCD}=1$ from the given condition and the fact that $\angle BAD+\angle BCD=180°$. This implies $AP=CP=\frac{56}{2}=28$.



Let $BP=x$ and $DP=65-x$. By the power chord theorem, we have $PA\times PC=PB\times PD$, or $28^2=x(65-x)$. This simplifies to $(x-16)(x-49)=0$. As $BC>DA$, $AP=CP$ and $\angle APD=\angle BPC$, we must have $BP>DP$. Thus, we reject $x=16$, leaving $x=49$ and $\frac{[ABC]}{[ADC]}=\frac{PB}{PD}=\frac{49}{16}$.

17. Note that $x^3+(2a+1)x^2+(4a-1)x+2=(x+2)(x^2+(2a-1)x+1)$. Hence all the roots of the equation are real if and only if the discriminant of the quadratic polynomial $x^2+(2a-1)x+1$ is non-negative, i.e. $(2a-1)^2-4\geq 0$. This is the same as $a\geq\frac{3}{2}$ or $a\leq-\frac{1}{2}$. Therefore, the probability that all the roots are real is $1-\frac{\frac{3}{2}-\left(-\frac{1}{2}\right)}{18-(-20)}=\frac{18}{19}$.

18. We divide the students into 3 groups — students 0, 3, 6, 9 belong to Group A, students 1, 4, 7, 10 belong to Group B and students 2, 5, 8, 11 belong to Group C.

Note that we can ignore students in group A since their scores must be multiples of 3. Suppose m students from Group B and n students from Group C obtained heads. Then we require $m+2n$ to be a multiple of 3 in order for the sum of all scores to be divisible by 3. That means we need $m\equiv -2n\equiv n$ (mod 3). We note that the probability for

- $m\equiv 0$ (mod 3), meaning $m=0$ or 3, is $\frac{C_0^4+C_3^4}{2^4}=\frac{5}{16}$;
- $m\equiv 1$ (mod 3), meaning $m=1$ or 4, is $\frac{C_1^4+C_4^4}{2^4}=\frac{5}{16}$;
- $m\equiv 2$ (mod 3), meaning $m=2$, is $\frac{C_2^4}{2^4}=\frac{6}{16}$.

and the same is true for n. Hence the required probability is $\frac{5}{16}\times\frac{5}{16}+\frac{5}{16}\times\frac{5}{16}+\frac{6}{16}\times\frac{6}{16}=\frac{43}{128}$.

19. Note that each jump can be one of $a=(5,0)$, $b=(0,5)$, $c=(3,4)$, $d=(4,3)$, $e=(3,-4)$, $f=(-4,3)$ or their negatives. The order of the jumps does not affect the final position. As each choice may be used more than once, the number of ways to choose 3 jumps from these 12 possibilities is $H_3^{12}=C_3^{12+3-1}=364$. It remains to remove from these 364 choices those that lead to the same final position.

First we consider the case in which two of the three jumps are negative of each other. In such cases the final position is one of $\pm a,\pm b,\pm c,\pm d,\pm e,\pm f$. Suppose final position is *a*. There are 6 ways to choose the routes since there are 6 ways to choose, in addition to *a*, a pair of jumps which cancel each other. That means 5 routes with final position *a* should be eliminated in our counting. Similarly, we need to eliminate 5 routes for each of the 12 final positions, making in total of $5\times 12=60$ routes to be eliminated.

Next, we consider those repetitions arisen from routes consisting of three jumps no two of which are negative of each other. Suppose $u+v+w=x+y+z$ where each variable corresponds to one of the 12 possible jumps. Then we can move everything to one side and eliminate like terms to obtain an expression of at most 6 terms with sum 0. Thus, it remains to find all ways to choose at most 6 terms from $\pm a,\pm b,\pm c,\pm d,\pm e,\pm f$ with sum 0.

Note that the *x*-coordinates of *a*, *c*, *e* are odd (call these 'odd terms'), while those of *b*, *d*, *f* are even (call these 'even terms'). Thus, there is an even number of odd terms. By symmetry, we may assume there are more odd than even terms.

- If there are 2 odd terms, then there are 2 even terms since the total number of terms is even. As the *y*-coordinates of the odd terms are multiples of 4, the only possibilities are $\pm(b+d)=\pm(4,8)$, $\pm(b+f)=\pm(-4,8)$ and $\pm(d-f)=\pm(8,0)$. None of these is a sum of two off terms.
- If there are 4 odd terms, then there are 2 even terms. Again, the only possibilities are $(4,8)$, $(-4,8)$ and $(8,0)$ up to a plus or minus sign. In order for the *y*-coordinate to be 8, we can only choose $3c+e=(12,8)$, $-c-3e=(-12,8)$, $\pm 2a+2c=(16,8),(-4,8)$, $\pm 2a+c-e=(\pm 10,8)$ or $\pm 2a-2e=(4,8),(-16,8)$. Among these, we obtain two identities $-2a-b+2c-f=0$ and $2a-b-2e-d=0$. Similarly, in order for the *x*-coordinate to be 8, we can only choose $a+2c-e=(8,12)$ or $a-c+2e=(8,-12)$. Thus we cannot obtain $(8,0)$.

It follows from symmetry that the only identities are $-2a-b+2c-f=0$, $2a-b-2e-d=0$, $-2b-a+2d-e=0$ and $2b-a-2f-c=0$. For each identity, we distribute the terms so that each side contains 3 terms, which corresponds to the original equation $u+v+w=x+y+z$. It is easy to check that there are exactly 10 ways to distribute the terms in each case. This shows that there are $10\times 4=40$ final positions which are counted twice.

By subtracting the repetitions, the number of distinct final positions is $364-60-40=264$.

20. The equation can be rewritten as $x^2+x+1+6(x-1)=5\sqrt{(x-1)(x^2+x+1)}$. Note that $x\neq 1$. This implies $\frac{x^2+x+1}{x-1}+6=5\sqrt{\frac{x^2+x+1}{x-1}}$. Setting $y=\sqrt{\frac{x^2+x+1}{x-1}}$, the equation becomes $y^2+6=5y$, or $(y-2)(y-3)=0$. It follows that y is either 2 or 3.

- If $y=2$, we square both sides of $\sqrt{\frac{x^2+x+1}{x-1}}=2$ and then simplify to obtain $x^2-3x+5=0$, which has no real solution.
- If $y=3$, we work on $\sqrt{\frac{x^2+x+1}{x-1}}=3$ similarly to obtain $x^2-8x+10=0$. The solutions $x=4\pm\sqrt{6}$ both satisfy $x>1$ and hence they are indeed solutions to the original equation.

The answer is thus the product of roots of the equation $x^2-8x+10=0$, which is 10.