香 港 資 優 教 育 學 苑
The Hong Kong Academy for Gifted Education

International Mathematical Olympiad
Preliminary Selection Contest – Hong Kong 2017

國際數學奧林匹克 — 香港選拔賽初賽 2017

20 May 2017 (Saturday)
2017 年 5 月 20 日（星期六）

Question Book
問題簿

Instructions to Contestants:
考生須知：

1. The contest comprises a 3 hours written test.
   比賽以筆試形式進行，限時三小時。

2. Questions are in bilingual versions. Answer all questions.
   題目中英對照。全卷題目均須作答。

3. Put your answers on the answer sheet.
   請將答案寫在答題紙上。

4. The use of calculators is NOT allowed.
   不可使用計算機。

5. Measuring instruments like rulers, compasses, etc. can be used.
   直尺、圓規及其它量度工具可作輔助之用。

Co-organised by The Hong Kong Academy for Gifted Education,
the Gifted Education Section of the Education Bureau and
International Mathematical Olympiad Hong Kong Committee
香港資優教育學苑、教育局資優教育組及國際數學奧林匹克香港委員會合辦

1. Let $a_0 = 1$, $a_1 = 2$ and $a_n = a_{n-2} + (a_{n-1})^2$ for $n > 1$. Find the remainder when a_{2017} is divided by 7. (1 mark)

設 $a_0 = 1$、$a_1 = 2$，且當 $n > 1$ 時皆有 $a_n = a_{n-2} + (a_{n-1})^2$。求 a_{2017} 除以 7 時的餘數。（1 分）

2. The graph of $x^2(x+y+1) = y^2(x+y+1)$ together with the two coordinate axes divide the xy-plane into n regions. Find the value of n. (1 mark)

$x^2(x+y+1) = y^2(x+y+1)$ 的圖像連同兩條座標軸把 xy-平面分成 n 個區域。求 n 的值。（1 分）

3. Let $f(n) = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots + \frac{1}{2017^n}$. Find the value of $f(2) + f(3) + f(4) + \cdots$. (1 mark)

設 $f(n) = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots + \frac{1}{2017^n}$。求 $f(2) + f(3) + f(4) + \cdots$ 的值。（1 分）

4. In how many different ways can we put 600 identical balls into 3 identical boxes such that each box contains at least one ball? (1 mark)

有多少種不同的方法可把 600 個全等的球放進 3 個全等的盒子，使得每個盒子中均最少有一個球？（1 分）

5. Let x_1, x_2, …, x_n be positive rational numbers such that $x_1{}^3 + x_2{}^3 + \cdots + x_n{}^3 = 1$. Find the smallest possible value of n. (1 mark)

設 x_1、x_2、…、x_n 為正有理數，使得 $x_1{}^3 + x_2{}^3 + \cdots + x_n{}^3 = 1$。求 n 的最小可能值。（1 分）

6. If b is a positive integer not exceeding 200, how many sets of solutions (a,b) are there to the equation $(\log_b a)^{2017} = \log_b(a^{2017})$? (1 mark)

若 b 是不超過 200 的正整數，方程 $(\log_b a)^{2017} = \log_b(a^{2017})$ 有多少組解 (a,b)？（1 分）

7. How many sets of positive integers (x, y, z) have the properties that $x \le y \le z$, their H.C.F. is 30 and their L.C.M. is 3000? (1 mark)

有多少組正整數 (x, y, z) 滿足以下各條件：$x \le y \le z$，它們的最大公因數為 30，且它們的最小公倍數為 3000？（1 分）

8. Let $f(x) = x^4 + x^3 + bx^2 + 100x + c$ and $g(x) = x^3 + ax^2 + x + 10$, where a, b, c are constants. If the equation $g(x) = 0$ has three distinct real roots, each of which is also a root of $f(x) = 0$, find the value of $f(1)$. (1 mark)

設 $f(x) = x^4 + x^3 + bx^2 + 100x + c$ 及 $g(x) = x^3 + ax^2 + x + 10$，其中 a、b、c 為常數。若方程 $g(x) = 0$ 有三個不同的實根，且當中每個都是 $f(x) = 0$ 的根，求 $f(1)$ 的值。（1 分）

9. In quadrilateral $ABCD$, $\angle BAD = \angle ADC$ and $\angle ABD = \angle BCD$. If $AB = 8$, $BD = 10$ and $BC = 6$, find the length of CD. (1 mark)

在四邊形 $ABCD$ 中，$\angle BAD = \angle ADC$ 及 $\angle ABD = \angle BCD$。若 $AB = 8$、$BD = 10$ 及 $BC = 6$，求 CD 的長度。 （1 分）

10. $ABCD$ is a trapezium with $\angle DAB = \angle ABC = 90°$ and $AB = BC = 2AD$. P is a point inside the trapezium such that $PA = 1$, $PB = 2$ and $PC = 3$. Find the area of $ABCD$. (1 mark)

$ABCD$ 是梯形，其中 $\angle DAB = \angle ABC = 90°$，且 $AB = BC = 2AD$。P 是梯形內的一點，使得 $PA = 1$、$PB = 2$ 及 $PC = 3$。求 $ABCD$ 的面積。 （1 分）

11. Ann and Ben compete in the election for the class club chairman. 8 eligible voters take turns to vote, and the vote count is updated immediately. A candidate wins once he/she leads the other by 4 votes during the process; otherwise there is no winner. If all voters cast a vote to one of Ann or Ben at random, what is the probability for Ann to win? (2 marks)

小芬和小明競逐班會主席一職。合資格投票者共有 8 人，他們逐一投票，之後票數會即時更新。過程中只要其中一位候選人領先另一人 4 票即告勝出，否則無人獲勝。若各人投票時均在小芬和小明之間隨機選擇一人，則小芬勝出的概率是多少？ （2 分）

12. Let d_1, d_2, …, d_n be all the positive factors of $11!$. Find the value of $\frac{1}{d_1+\sqrt{11!}}+\frac{1}{d_2+\sqrt{11!}}+\cdots+\frac{1}{d_n+\sqrt{11!}}$. (2 marks)

設 d_1、d_2、…、d_n 為 $11!$ 的全部正因數。求 $\frac{1}{d_1+\sqrt{11!}}+\frac{1}{d_2+\sqrt{11!}}+\cdots+\frac{1}{d_n+\sqrt{11!}}$ 的值。 （2 分）

13. If all roots of the equation $(k^2-1)x^2-6(3k-1)x+72=0$ are positive integers, find the sum of all possible values of the constant k. (2 marks)

若方程 $(k^2-1)x^2-6(3k-1)x+72=0$ 的所有根皆是正整數，求常數 k 的所有可能值之和。 （2 分）

14. Let $Q(x)=x^2+ax+b$, where a and b are integers with $|b|\le 365$. If the equation $[Q(x)]^2=1$ has four positive integral solutions (not necessarily distinct), how many different sets of possible values for (a,b) are there? (2 marks)

設 $Q(x)=x^2+ax+b$，其中 a、b 為整數，且 $|b|\le 365$。若方程 $[Q(x)]^2=1$ 有四個正整數解（不一定相異），則 (a,b) 共有多少組不同的可能值？ （2 分）

15. In ΔABC, D and E are points on AB and AC respectively. If $AB = 33$, $AC = 21$, $BC = m$ and $AD = DE = EC = n$ where m, n are integers, find the value of m. (2 marks)

在 ΔABC 中，D 和 E 分別為 AB 和 AC 上的點。若 $AB = 33$、$AC = 21$、$BC = m$ 且 $AD = DE = EC = n$（其中 m、n 為整數），求 m 的值。 （2 分）

16. Some families form teams to join a competition. Each family consists of the father, the mother and not more than five children. There is a prize each for the best father, the best mother and the best child, with the restriction that each family may win at most once prize. It turns out that there are 7770 different possible combinations for the prize winners. What is the total number of children joining the competition? (2 marks)

某比賽中，參加者以家庭為單位組隊，每個家庭由父親、母親和不超過五名孩子組成。比賽設最佳父親、最佳母親和最佳孩子獎項各一，但規定每個家庭最多只能獲獎一次。若得獎者的可能組合共有 7770 個，則參賽的孩子總數是多少？（2 分）

17. In ΔABC, $AB=4$ and $AC=6$. The two tangents to the circumcircle of ΔABC at B and C intersect at D. If A and D are equidistant from the line BC, find the length of BC. (2 marks)

在 ΔABC 中，$AB=4$ 及 $AC=6$。ΔABC 的外接圓於 B 和 C 的兩條切線相交於 D。若 A 和 D 到直線 BC 的距離相等，求 BC 的長度。（2 分）

18. On the plane 5 points X_1, X_2, …, X_5 are chosen, and each pair of these points are joined by a blue line. Suppose no two blue lines are parallel or perpendicular to each other. Now for each point X_i $(1 \leq i \leq 5)$ and each blue line ℓ not passing through X_i, a red line is drawn passing though X_i and perpendicular to ℓ. What is the maximum number of points of intersection formed by the red lines? (2 marks)

現於平面上選 5 點 X_1、X_2、⋯、X_5，且它們每兩點之間均以一條藍線連起。假設沒有兩條藍線互相平行或垂直。對於每點 X_i（$1 \leq i \leq 5$）和每條不穿過 X_i 的藍線 ℓ，我們均構作一條穿過 X_i 且垂直於 ℓ 的紅線。那麼，由紅線組成的交點最多有幾個？（2 分）

19. Let ABC be an acute-angled triangle with incircle ω which is tangent to sides BC and CA at D and E, respectively. X is a point on the altitude from A to BC, and the circle ω' with diameter AX is tangent to ω. Denote by U and V, respectively, the points where CA and AB intersect ω' again. If $UV=12$, $AX=15$ and $AE=24$, find the value of $BD \times DC$. (2 marks)

設 ABC 為銳角三角形，ω 為其內切圓，且 ω 分別與邊 BC 和 CA 相切於 D 和 E。X 是 A 到 BC 的高上的一點，而以 AX 為直徑的圓 ω' 與 ω 相切。分別以 U 和 V 表示 CA 和 AB 與 ω' 的另一交點。若 $UV=12$、$AX=15$ 及 $AE=24$，求 $BD \times DC$ 的值。（2 分）

20. Let X be a randomly chosen subset of $\{1, 2, \ldots, 2017\}$. A positive integer n is said to be 'good' if both n and $n+|X|$ are elements of X (where $|X|$ denotes the number of elements of X). Find the expected value of the number of 'good' integers. (2 marks)

現從 $\{1, 2, \ldots, 2017\}$ 隨機選出一個子集 X。對於正整數 n，若 n 和 $n+|X|$（其中 $|X|$ 表示 X 的元素的數目）均為 X 的元素，則 n 稱為「好數」。求「好數」的數量的期望值。（2 分）

– End of Paper – 全卷完 –