

**International Mathematical Olympiad**  
**Preliminary Selection Contest 2016 — Hong Kong**

**Outline of Solutions**

**Answers:**

- |                       |                     |                           |                   |
|-----------------------|---------------------|---------------------------|-------------------|
| 1. 28224              | 2. $\frac{1}{4}$    | 3. 72                     | 4. 109            |
| 5. 4063248            | 6. 5                | 7. $\frac{\sqrt{473}}{7}$ | 8. 22             |
| 9. 394                | 10. 12504           | 11. $70^\circ$            | 12. 48384         |
| 13. 113               | 14. $\sqrt[3]{290}$ | 15. 1132                  | 16. $\frac{3}{4}$ |
| 17. $\frac{19}{2187}$ | 18. $\frac{96}{25}$ | 19. 203                   | 20. $\frac{1}{6}$ |

**Solutions:**

1. Since  $2016 = 2^5 \times 3^2 \times 7$ ,  $n$  is of the form  $2^{2a} \times 3^{2b} \times 7^{2c} \times d^2$  where  $2a \geq 5$ ,  $2b \geq 2$  and  $2c \geq 1$ . The smallest possible value of  $n$  is  $2^6 \times 3^2 \times 7^2 = 28224$ .
2. There are three possibilities for the leftmost digit (1, 2 or 6) and four possibilities for the rightmost digit (0, 1, 2 or 6). Each of these  $3 \times 4 = 12$  combinations is equally likely to occur. There are three favourable outcomes for the leftmost and rightmost digits, namely, (1,2), (1,6) and (2,6). Hence the answer is  $\frac{3}{12} = \frac{1}{4}$ .
3. Let the interior angles be  $a^\circ$ ,  $b^\circ$  and  $c^\circ$  in ascending order. Then we have  $c = a + 54$  and  $b = 180 - a - (a + 54) = 126 - 2a$ . As  $b$  lies between  $a$  and  $c$ , we have  $a \leq 126 - 2a \leq a + 54$ . Solving gives  $24 \leq a \leq 42$ .

To maximise  $x - y$ , we should maximise  $x$  and minimise  $y$ . To maximise  $x$ , we add up  $b$  and  $c$  with  $a$  as small as possible, giving  $x = b + c = 180 - a \leq 180 - 24 = 156$ . To minimise  $y$ , we add

up  $a$  and  $b$  and, since  $a+b=126-a$ , we seek to make  $a$  as large as possible, giving  $y = a+b = 126-a \geq 126-42 = 84$ . (Note also that both 156 and 84 are indeed feasible, with the former corresponding to  $a = 24$  and  $b = c = 78$  and the latter corresponding to  $a = b = 42$  and  $c = 96$ .)

It follows that the answer is  $156 - 84 = 72$ .

4. The total score is at least  $0+1+2+\dots+25 = 325$ . Since the students altogether get at most 3 marks for each question, there are at least  $\left\lceil \frac{325}{3} \right\rceil = 109$  questions.

It is not hard to construct an example with  $n = 109$ . For example, we number the students 0, 1, 2, ..., 25, each of whom gets a score equal to their number. They take turns (starting with student 25 backward) to pick the lowest-numbered available questions to be the questions they answer correctly subject to each question being picked at most three times. (For instance student 25 answers Questions 1 to 25, student 24 answers Questions 1 to 24, student 23 answers Questions 1 to 23, student 22 answers Questions 24 to 45, student 21 answers Questions 25 to 45 and so on. In this way one can easily check that 109 questions are sufficient.)

It follows that the answer is 109.

5. We have  $f(n) = \left\lfloor \frac{4032n}{2017} \right\rfloor = \left\lfloor 2n - \frac{2n}{2017} \right\rfloor = 2n + \left\lfloor -\frac{2n}{2017} \right\rfloor$ . For  $1 \leq n \leq 1008$ , we have  $\left\lfloor -\frac{2n}{2017} \right\rfloor = -1$ . For  $1009 \leq n \leq 2016$ , we have  $\left\lfloor -\frac{2n}{2017} \right\rfloor = -2$ . Hence the required sum is equal to  $(1+3+5+\dots+2015) + (2016+2018+\dots+4030) = 4031 \times 1008 = 4063248$ .

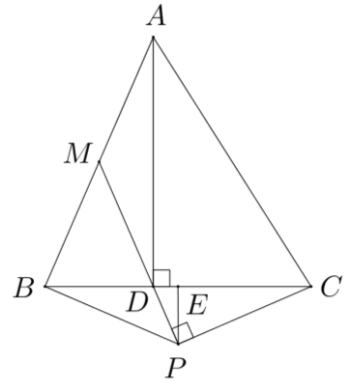
6. Through expansion and rearranging terms, we find that

$$4x^2 + (x+2y-6)^2 + 16y - 23 = (x+2y-2)^2 + 4(x-1)^2 + 5.$$

It follows that the smallest possible value is 5, which can be attained when  $x = 1$  and  $y = \frac{1}{2}$ .

**Remark.** For those who know calculus, one could simply set the first order partial derivatives to zero and solve for the corresponding values of  $x$  and  $y$ .

7. Let  $E$  be the mid-point of  $BC$ . As  $PB = PC$ , we have  $PE \perp BC$ . Since  $M$  is the mid-point of  $AB$  and  $\angle ADB = 90^\circ$ , we have  $MB = MD$ . From this, we get  $\angle MBD = \angle MDB = \angle PDE$ . Thus  $\triangle PED \sim \triangle ADB$ . Note that we also have  $\triangle PED \sim \triangle CPD$ . Let  $DE = x$ . From the similar triangles and the fact that  $AB = 3BD$ , we have  $DP = 3x$  and  $CD = 9x$ . Hence  $BE = CE = 9x - x = 8x$  and  $BD = 8x - x = 7x$ . This yields  $x = \frac{1}{7}$ . It follows that  $CD = \frac{9}{7}$



and so  $AC = \sqrt{AD^2 + CD^2} = \sqrt{AB^2 - BD^2 + CD^2} = \frac{\sqrt{473}}{7}$ .

**Remark.** This problem may also be solved using coordinate geometry as follows. Set the origin at  $D$ , and let  $B = (-1, 0)$ ,  $A = (0, 2\sqrt{2})$  and  $C = (c, 0)$ . Then  $M = \left(-\frac{1}{2}, \sqrt{2}\right)$  and hence the equation of  $MD$  is  $y = -2\sqrt{2}x$ . The equation of  $CP$  is thus  $y = \frac{1}{2\sqrt{2}}x - \frac{c}{2\sqrt{2}}$ . Solving these, we get  $P = \left(\frac{c}{9}, -\frac{2\sqrt{2}c}{9}\right)$ . In order that  $PB = PC$ , the  $x$ -coordinate of  $P$  should be the mean of those of  $B$  and  $C$ , i.e.  $\frac{c-1}{2} = \frac{c}{9}$ . This gives  $c = \frac{9}{7}$ , from which  $AC = \frac{\sqrt{473}}{7}$  follows.

8. By the sum of roots, we have  $c + d = -a$  and  $a + b = -c$ . These give  $b = d$ . If  $b = d = 0$ , then  $c = -a$  and the conditions are satisfied. There are 21 such cases, corresponding to the 21 choices of  $c$  between  $-10$  and  $10$ .

If  $b = d \neq 0$ , then by the product of roots, we have  $cd = b$  and  $ab = d$ . Hence  $a = c = 1$ . From the sum of roots relations above, we get  $b = d = -2$ , giving rise to just one set of possible values for  $a, b, c$  and  $d$ .

It follows that the answer is 22.

9. Since  $Q(1) = 4$ , we know that each coefficient of  $Q$  can only be 0, 1, 2, 3 or 4. Thus the coefficients are precisely the digits in the base 5 representation of  $Q(5)$ . As  $152 = 5^3 + 5^2 + 2$ , i.e.  $152_{(10)} = 1102_{(5)}$ , we have  $Q(x) = x^3 + x^2 + 2$  and so  $Q(7) = 7^3 + 7^2 + 2 = 394$ .
10. There are 90000 five-digit positive integers, among which 30000 of them are multiples of 3. (To see this, note that with the last four digits fixed, for example 1234, the sum of digits of 11234, 21234, ..., 91234 are nine consecutive positive integers, so exactly three of them are divisible by 3.) We need to subtract the multiples of 3 which do not contain the digit '3'.

To count such numbers for subtraction, note that there are 8 choices for the first digit (all

except 0 and 3), and then 9 choices for each of the thousands, hundreds and tens digits (all except 3). The unit digit can be one of 0, 1, 2, 4, 5, 6, 7, 8, 9, but for the number to be divisible by 3, exactly one of 0, 1 and 2 will work using the same reasoning as in the previous paragraph, and similarly exactly one of 4, 5 and 6, and also exactly one of 7, 8 and 9 will work. Hence a total of  $8 \times 9 \times 9 \times 9 \times 3 = 17496$  numbers need to be removed.

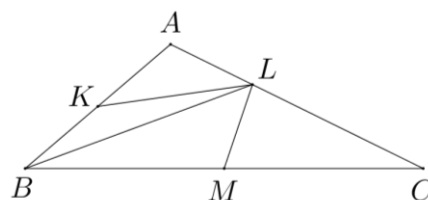
It follows that the answer is  $30000 - 17496 = 12504$ .

11. By the angle bisector theorem, we have  $\frac{AL}{LC} = \frac{AB}{BC}$ .

Note that  $AL + LC = AC$  while  $AB + BC = 2AC$ . It follows that  $AL = \frac{1}{2}AB = AK$  and  $LC = \frac{1}{2}BC = CM$ .

Thus  $\triangle AKL$  is isosceles with  $\angle ALK = \frac{180^\circ - A}{2}$ , and

similarly  $\angle CLM = \frac{180^\circ - C}{2}$ . As a result, we have



$$\angle KLM = 180^\circ - \frac{180^\circ - A}{2} - \frac{180^\circ - C}{2} = \frac{A + C}{2} = \frac{180^\circ - B}{2} = 70^\circ.$$

12. Note that  $2016 = 2^5 \times 3^2 \times 7$ . Considering mod 4, the last two digits must be 44, 84, 48 or 88. For the first two cases, the hundreds digit may be 3, 5, 7, 9 if we consider mod 8, while for the last two cases the hundreds digit may be 4 or 8.

Now we take mod 16 and mod 9 into account. For instance, if the first last three digits are 344, then since 344 is not divisible by 16, the thousands digit has to be odd. Mod 9 forces the number to be 7344, 25344, 43344, ..., with possible values in increments of 18000. Of course, some (e.g. 25344) can be discarded owing to the existence of '2', '0', '1' or '6'.

Repeating this check, we can list the potential candidates after mod 16, mod 9 and the restrictions on '2', '0', '1', '6' have been considered. In ascending order, we have

3744, 3888, 7344, 7488, 8784, 33984, 34848, 37584, 38448, 39744, 39888, 43344, 43488, 44784, ...

One can check that almost all numbers in the above list fail the mod 7 test. For example 3744 is not divisible by 7 since  $744 - 3$  is not. The only exception is 43344 as  $344 - 43 = 7 \times 43$ . However it is not divisible by 32.

The next one on the list (not shown above) is 48384. It is divisible by 7 since  $384 - 48 = 7 \times 48$ , and divisible by 32 since  $48384 - 32000 = 16384 = 2^{14}$ . Hence this is the number we want.

13. If  $OB = AC$ , then  $OACB$  is a rectangle. Every point  $D$  on segment  $OA$  has the property that the area of  $\triangle BDC$  is equal to the sum of the areas of  $\triangle ODB$  and  $\triangle CDA$ . Hence, the only constraint is that  $OA + OB = 16$  and  $0 < OD < OA$ . If  $OA = n$  (where  $n$  could be 1, 2, ..., 15), there will be  $n - 1$  choices for the position of  $D$  (e.g. if  $OA = 5$  we could choose the length of  $OD$  to be 1, 2, 3 or 4). This gives  $1 + 2 + \dots + 14 = 105$  sets of possible answers.

Now suppose  $OB \neq AC$ . The area condition  $\frac{OB \times OD}{2} + \frac{AC \times AD}{2} = \frac{(OB + AC) \times OA}{4}$  implies  $OD = AD$ , i.e.  $D$  must be the midpoint of  $AO$ , and so  $AO$  is even. By Pythagoras' theorem, we have  $OA^2 + (OB - AC)^2 = BC^2$ . Hence, we consider the positive integer solutions to  $a^2 + b^2 = c^2$  with  $a + b + c < 32$ . For the solutions (3, 4, 5) and (5, 12, 13),  $OA$  must correspond to the only even length. There are 2 cases corresponding to each solution since the lengths of  $OB$  and  $AC$  can be interchanged. For the solution (6, 8, 10),  $OA$  can be either 6 or 8, and each corresponds to 2 possible configurations. Thus there are  $2 + 2 + 2 + 2 = 8$  sets of possible answers in this case.

Hence the answer is  $105 + 8 = 113$ .

14. Note that  $(a^2 + b^2)^3 = (a^3 - 3ab^2)^2 + (b^3 - 3a^2b)^2$ . Hence  $(a^2 + b^2)^3 = 11^2 + 13^2 = 290$  and so the answer is  $\sqrt[3]{290}$ .
15. As there are two types of 'perfect groups', we consider the following cases according to the number of groups of each type. Note that if a 'perfect group' is of the form  $XXX$ , then  $X$  could be between 1 and 10, while if it is of the form  $(X-1)X(X+1)$ , then  $X$  is between 2 and 9.

Case 1: The 'perfect groups' are AAA, BBB and CCC

We need to choose A, B, C (possibly the same) out of 1 to 10. There are  $H_3^{10} = C_3^{10+3-1} = 220$  choices.

Case 2: The 'perfect groups' are AAA, BBB and (C-1)C(C+1)

Each of A and B (possibly the same) is chosen between 1 and 10 while C is chosen between 2 and 9. Thus there are  $H_2^{10} \times 8 = 440$  choices in this case.

Case 3: The 'perfect groups' are AAA, (B-1)B(B+1) and (C-1)C(C+1)

A is between 1 and 10 while B and C (possibly the same) are between 2 and 9. Thus there are  $10 \times H_2^8 = 360$  choices in this case.

Case 4: The 'perfect groups' are (A-1)A(A+1), (B-1)B(B+1) and (C-1)C(C+1)

There are  $H_3^8 = 120$  ways to choose these numbers.

However, if each of  $(A-1)$ , A and  $(A+1)$  appear three times, such combinations are counted in

both Case 1 and Case 4. There are 8 such cases, corresponding to  $A$  being 2, 3, 4, ..., 9. Hence the answer is  $220+440+360+120-8=1132$ .

16. As usual, let  $a, b, c$  denote the lengths of the sides opposite  $A, B, C$  respectively. It is given that  $a+c=2b$ . In view of the sine formula, this implies  $\sin A+\sin C=2\sin B$ , i.e.

$$\sin \frac{A+C}{2} \cos \frac{A-C}{2} = 2 \sin \frac{B}{2} \cos \frac{B}{2}.$$

Note that we have  $\cos \frac{A-C}{2} = \cos 45^\circ = \frac{1}{\sqrt{2}}$  and  $\sin \frac{A+C}{2} = \sin \frac{180^\circ-B}{2} = \cos \frac{B}{2} \neq 0$ . It

follows that  $\sin \frac{B}{2} = \frac{1}{2\sqrt{2}}$  and so  $\cos B = 1 - 2\sin^2 \frac{B}{2} = \frac{3}{4}$ .

17. For  $n \geq 2$ , let  $a_n$  be the number of ways of putting on hats for which the condition is satisfied when there are  $n$  boys instead of 10. We have  $a_2 = a_3 = 3$  since the colours of the hats of all boys must be the same in these cases. Now we try to count  $a_n$  with  $n \geq 4$ :

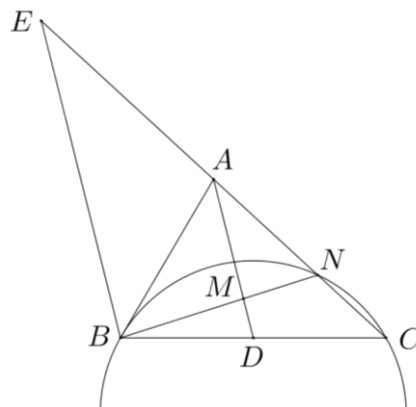
- Note that the first boy and the second boy must put on hats of the same colour.
- If the colour of the hat of the third boy is also the same, then by considering the first and the second boy as a whole, there are  $a_{n-1}$  ways to assign hats for the boys.
- If the colour of hat of the third boy is different from the first two boys, then there are  $a_{n-2}$  ways to assign hats for the last  $n-2$  boys, and then two ways for the first two boys (they both put on hats of the same colour which is different from the third boy's).

It follows that we have the recurrence relation  $a_n = a_{n-1} + 2a_{n-2}$ . Using this and the initial conditions  $a_2 = a_3 = 3$ , we can work out the values of the terms of  $\{a_n\}$  as follows: 3, 3, 9, 15, 33, 63, 129, 255, 513, ... With  $a_{10} = 513$  and  $3^{10}$  ways of putting on hats without the condition, the required probability is thus  $\frac{513}{3^{10}} = \frac{19}{2187}$ .

18. From the tangent, we have  $\triangle ABN \sim \triangle ACB$ . Thus  $\frac{AB}{AC} = \frac{AN}{AB} = \frac{BN}{CB} = \frac{3}{4}$ . This gives  $\frac{AN}{AC} = \frac{AB}{AC} \times \frac{AN}{AB} = \frac{9}{16}$ .

Let  $E$  be the intersection of  $AC$  and the line through  $B$  parallel to  $AD$ . Since  $BD = DC$ , we have  $EA = AC$ . It follows that  $\frac{BM}{MN} = \frac{EA}{AN} = \frac{AC}{AN} = \frac{16}{9}$ . Together with

$BM + MN = BN = 6$ , we get  $BM = 6 \times \frac{16}{16+9} = \frac{96}{25}$ .



19. Note that for any integers  $m$  and  $n$ , the integer  $m-n$  must divide  $f(m)-f(n)$  since  $f$  is a polynomial with integer coefficients. In particular, 4 divides  $f(4)-f(0)$  and 3 divides  $f(3)-f(0)$ . Considering mod 4, we see that  $f(0)$  must be one of 20 and 2016, while  $f(4)$  takes the other value. Taking mod 3 into account, we see that the only possible cases are  $(f(0), f(3), f(4)) = (20, 2, 2016)$  and  $(2016, 201, 20)$ . These two cases correspond to  $f(x) = 505x^2 - 1521x + 20$  and  $f(x) = 106x^2 - 923x + 2016$  respectively. The only possible values of  $f(1)$  are thus  $-996$  and  $1199$  respectively. Their sum is 203.

20. Let  $BP = x$  and  $CQ = y$  so that  $AP = 8x$  and  $AQ = 15y$ . Then  $AB = 9x$  and  $AC = 16y$ . By the power chord theorem, we have  $BX = \sqrt{BP \times BA} = 3x$  and  $CY = \sqrt{CQ \times CA} = 4y$ .

Note also that  $BA = BY$  and  $CA = CX$ . Hence we have  $BA - BX = CA - CY$  (both being equal to  $XY$ ), or  $9x - 3x = 16y - 4y$ . This gives  $x = 2y$ .

It follows that  $AB = 18y$  and  $BC = BX + XC = 3x + 16y = 22y$ . Hence  $AB : BC : CA = 9 : 11 : 8$

and so the cosine formula implies  $\cos \angle BAC = \frac{9^2 + 8^2 - 11^2}{2 \times 9 \times 8} = \frac{1}{6}$ .

