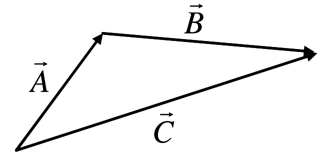
Mechanics

1 Vector

1.1

Any vector $\vec{A} = A_x\vec{x}_0 + A_y\vec{y}_0 + A_z\vec{z}_0$ (1.1)

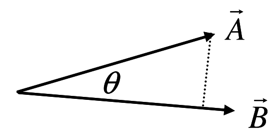


Addition of two vectors:

$\vec{C} = \vec{A} + \vec{B} = (A_x + B_x)\vec{x}_0 + (A_y + B_y)\vec{y}_0 + (A_z + B_z)\vec{z}_0$ (1.2)

Dot product of two vectors

$\vec{A} \bullet \vec{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$ (1.3)



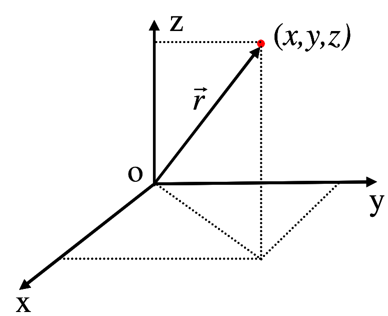
It is also referred to as the projection of $\vec{A}$ on $\vec{B}$, or vise versa.

Amplitude of the vector

$|\vec{A}| \equiv \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{\vec{A} \bullet \vec{A}}$ (1.4)

1.2

Position vector of a particle: $\vec{r} = x\vec{x}_0 + y\vec{y}_0 + z\vec{z}_0$ (1.5).



If the particle is moving, then *x, y,* and *z* are function of time *t*.

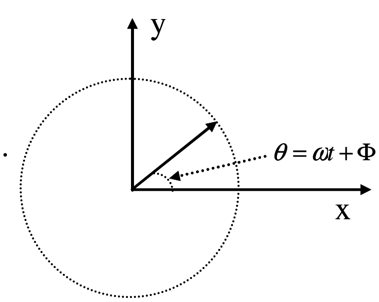
Velocity:

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{x}_0 + \frac{dy}{dt}\vec{y}_0 + \frac{dz}{dt}\vec{z}_0 = v_x\vec{x}_0 + v_y\vec{y}_0 + v_z\vec{z}_0 \quad (1.6)$$

Acceleration

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\vec{x}_0 + \frac{dv_y}{dt}\vec{y}_0 + \frac{dv_z}{dt}\vec{z}_0 = \frac{d^2x}{dt^2}\vec{x}_0 + \frac{d^2y}{dt^2}\vec{y}_0 + \frac{d^2z}{dt^2}\vec{z}_0 = a_x\vec{x}_0 + a_y\vec{y}_0 + a_z\vec{z}_0 \quad (1.7)$$

1.3 Uniform circular motion



Take the circle in the X-Y plane, so $z = 0$,

$x = R\cos(\omega t + \Phi)$, $y = R\sin(\omega t + \Phi)$ (1.8)

ω is the angular speed. Φ is the initial phase. Both are constants.

Using the above definition of velocity (1.6),

$\vec{v} \equiv -R\omega\sin(\omega t + \Phi)\vec{x}_0 + R\omega\cos(\omega t + \Phi)\vec{y}_0$ (1.9),

$\vec{v}$ is always perpendicular to $\vec{r}$.

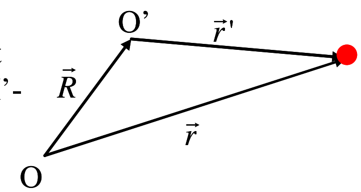
Its amplitude is $v = R\omega$ (1.10)

The acceleration is:

$\vec{a} \equiv -R\omega^2\cos(\omega t + \Phi)\vec{x}_0 - R\omega^2\sin(\omega t + \Phi)\vec{y}_0 = -\omega^2\vec{r}$ (1.11).

Its amplitude is $a = R\omega^2 = \frac{v^2}{R}$ (1.12)

2 **Relative Motion**



A reference frame is needed to describe any motion of an object. Consider two such reference frames S and S' with their origins at O and O', respectively. The X-Y-Z axes in S are parallel to the X'-Y'-Z' axes in S'. One is moving relative the other.

Note: $\vec{r} = \vec{r}' + \vec{R}$. (2.1)

So the velocity is: $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \frac{d\vec{R}}{dt} = \vec{v}' + \vec{u}$ (2.2)

Similar for acceleration: $\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d\vec{v}'}{dt} + \frac{d\vec{u}}{dt} = \vec{a}' + \vec{A}$ (2.3)

This is the classic theory of relativity. If $\vec{u}$ is constant, then $\vec{A} = 0$, and $\vec{a} = \vec{a}'$, i. e., Newton's Laws work in all inertia reference frames.
Properly choosing a reference frame can sometimes greatly simplify the problems.

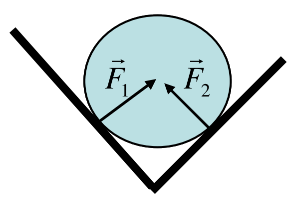
3 **Forces**

Pull through a rope, push, contact forces (elastic force and friction force), air resistance, fluid viscosity, surface tension of liquid and elastic membrane, *gravity, electric and magnetic, strong interaction, weak interaction*. Only the last four are fundamental. All the others are the net effect of the *electric and magnetic* force.

3.1 Tension

Pulling force (tension) in a thin and light rope:
Two forces, one on each end, act along the rope direction. The two forces are of equal amplitude and opposite signs because the rope is massless. It is also true for massless sticks.

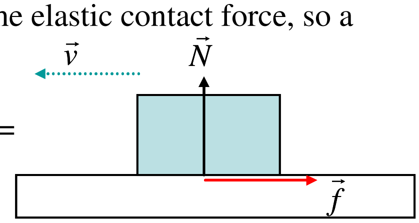
3.2 Elastics



Elastic contact forces are due to the deformation of solids. Usually the deformation is so small that it is not noticed. The contact force is always perpendicular to the contact surface. In the example both $\vec{F}_1$ and $\vec{F}_2$ are pointing at the center of the sphere.

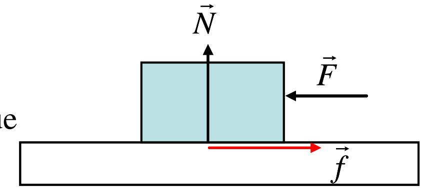
3.3 Friction

The friction force between two contact surfaces is caused by the relative motion or the *tendency* of relative motion. Its amplitude is proportional to the elastic contact force, so a friction coefficient μ can be defined.



When there is relative motion, the friction force is given by $f = \mu_k N$, where μ_k is the kinetic friction coefficient. The direction of the friction is always opposite to the direction of the relative motion.

When there is no relative motion but a tendency for such motion, like a block is being pushed by a force $\vec{F}$, the amplitude of f is equal to F until it reaches the limiting value of $\mu_s N$ if F keeps increasing, where μ_s is the static friction coefficient.



Once the block starts to move, the friction becomes $f = \mu_k N$. Usually $\mu_k < \mu_s$.

3.4 Viscosity

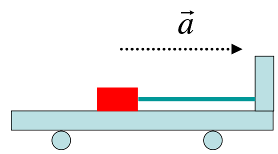
Air resistance and fluid viscous forces are proportional to the relative motion speed and the contact area. A coefficient called viscosity γ is used in these cases.

3.5 Inertial force

Inertial force is a 'fake' force which is present in a reference frame (say S') which itself is accelerating. Recall that $\vec{a} = \vec{a}' + \vec{A}$. Assume frame-S is not accelerating, then according to Newton's Second Law, $\vec{F} = m\vec{a} = m(\vec{a}' + \vec{A})$. So in the S'-frame, if one wants to correctly apply Newton's Law, she will get $\vec{F} - m\vec{A} = m\vec{a}'$, i. e., there seems to be an additional force

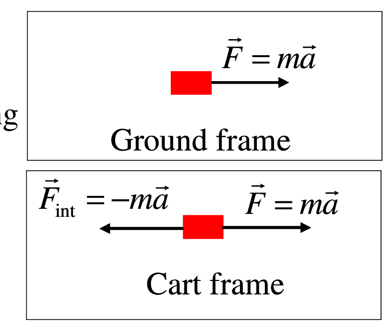
$$\vec{F}_{\text{int}} = -m\vec{A} \qquad (3.1)$$

acting upon the object.



Example 3.1

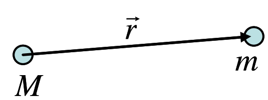
A block is attached by a spring to the wall and placed on the smooth surface of a cart which is accelerating. According to the ground frame, the force $\vec{F}$ acting on the block by the spring is keeping the block accelerating with the cart, so $\vec{F} = m\vec{a}$. In the reference frame on the cart, one sees the block at rest but there is a force on the block by the spring. This force is 'balanced' by the inertial force $\vec{F}_{\text{int}} = -m\vec{a}$



3.6 Gravity

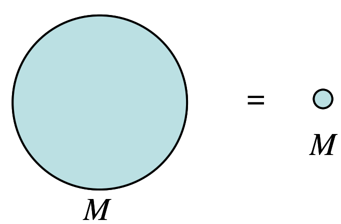
Between two point masses M and m, the force is

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \qquad (3.2)$$



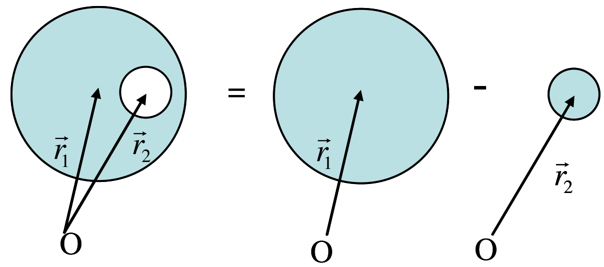
The gravitation field due to M is $\vec{g} = -\frac{GM}{r^2}\hat{r}$ (3.3).

The field due to a sphere at any position outside the sphere is equal to that as if all the mass is concentrated at the sphere center. (Newton spent nearly 10 years trying to proof it.)



Potential energy $U = \frac{-GM}{r}$ (3.4).

Application of superposition: Uniform density larger sphere with a smaller spherical hole.

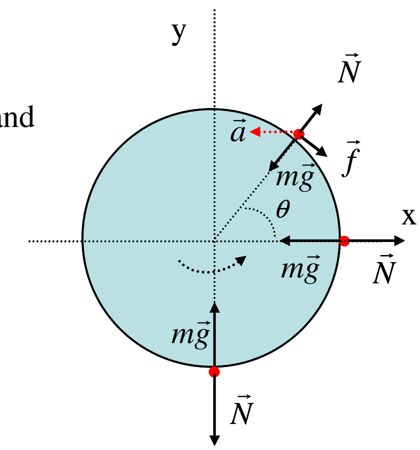


Near Earth surface, because the large radius of Earth, the gravitation field of Earth can be taken as constant, and its amplitude is

$$g = \frac{GM_E}{R_E^2} = 9.8 \text{ m/s}^2 \qquad (3.5).$$

Its direction is pointing towards the center of Earth, which in practice can be regarded as 'downwards' in most cases.

Example 3.2



The 'weight', or the force of the ground on a person on different places on Earth. Take the radius of Earth as R, and the rotation speed being ω (= 2π/86400 s^{-1}).
On the Equator, we have $mg - N = ma = m\omega^2 R$,
so $N = mg - m\omega^2 R = m(g - \omega^2 R)$

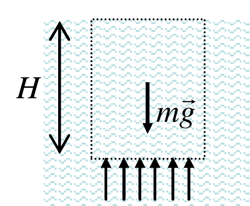
At the South Pole (or North Pole), we have $N = mg$
At latitude θ, by breaking down the forces along the X-direction and Y-direction, we have

$(mg - N)\cos\theta - f\sin\theta = ma$, and
$(mg - N)\sin\theta + f\cos\theta = 0$.

Solving the two equations, we get $(mg - N) = ma\cos\theta$, $f = -ma\sin\theta$, where $a = \omega^2 R\cos\theta$.
The negative sign of f means that its direction is the opposite of what we have guessed.

One can also break down the forces along the tangential and radial directions to obtain the same answers. One can also take Earth as reference frame and introduce the inertia force to account for the rotational acceleration.

3.7 Buoyancy

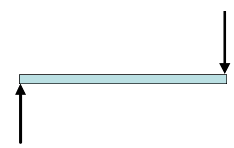


In a fluid (liquid or gas) of mass density ρ at depth H, consider a column of it with cross section area A, then the total mass of the column is ρAH, and the gravity on it is ρAHg. The gravity must be balanced by the supporting force from below, so the force of the column on the rest of the liquid is $F = \rho AHg$ and pointing downwards. The pressure

$P = F/A = \rho Hg$ (3.6).

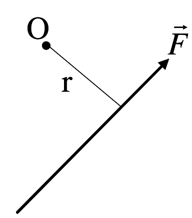
Now consider a very small cubic of fluid with all six side area of A at depth H. The force on its upper surface is ρAHg and pointing down, the force on its lower surface is ρAHg but pointing upwards so the cubic is at rest. However, for the cubic not to be deformed by the two forces on its upper and lower surfaces, the forces on its side surfaces must be of the same magnitude. This leads to the conclusion that the pressure on any surface at depth H is ρAHg, and its direction is perpendicular to the surface. One can then easily prove that the net force of the fluid (*buoyancy*) on a submerged body of volume V is equal ρVg. (See the HKPhO 2003 paper.) The buoyancy force is acting on the center of mass of the *submerged portion* of the object.

3.8 Torque

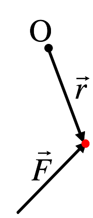


When two forces of equal amplitude and opposite directions acting upon the two ends of a rod, the center of the rod remains stationary but the rod will spin around the center. The torque (of a force) is introduced to describe its effect on the rotational motion of the object upon which the force is acting. First, an origin (pivot) point O should be chosen. The amplitude of the torque of force $\vec{F}$ is

$$\tau = rF \qquad (3.7),$$



where r is the distance between $\vec{F}$ and the origin O. The direction of the torque (a vector as well) is point out of the paper surface using the right hand rule. *One can choose any point as origin*, so the torque of a force depends on the choice of origin. However, for two forces of equal amplitude and opposite directions, the total torque is independent of the origin.



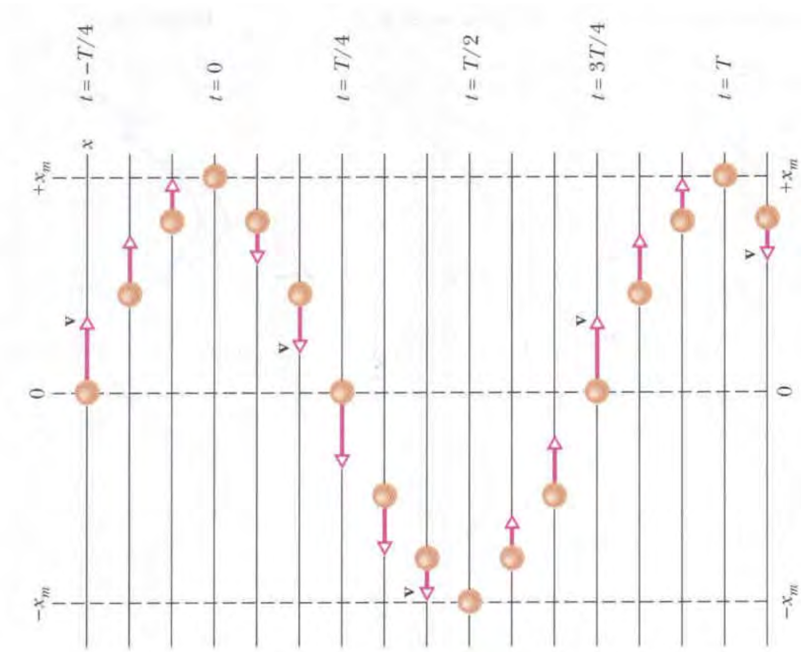
The general form of torque is defined as

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \qquad (3.8),$$

which involves the cross-product of two vectors.

4 Oscillations

4.1 Simple Harmonic Motion



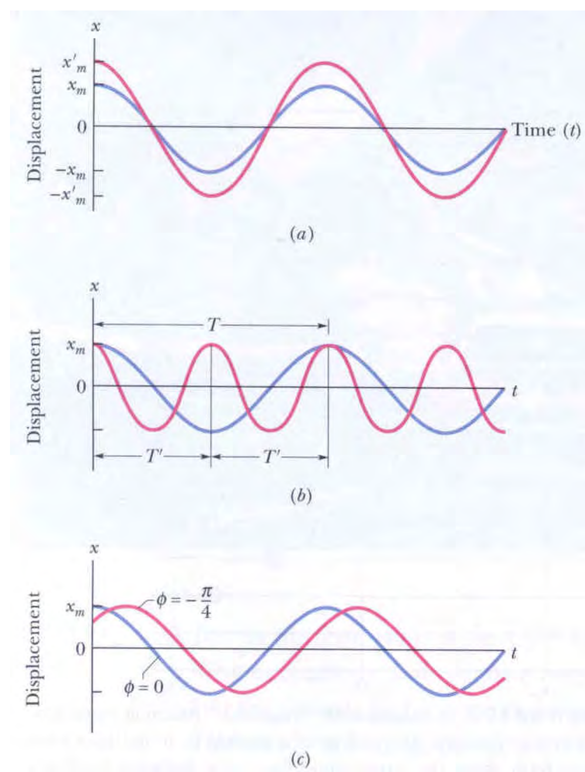
Frequency f and Period T: $\quad T = \dfrac{1}{f}$

$$x(t) = x_m \cos(\omega t + \phi) \qquad (4.1)$$

(a) Effects of different amplitudes

(b) Effects of different periods

(c) Effects of different phases



Since the motion returns to its initial value after one period T,

$$x_m \cos(\omega t+\phi) = x_m \cos[\omega(t+T)+\phi],$$
$$\omega t+\phi+2\pi = \omega(t+T)+\phi,$$
$$\omega T = 2\pi.$$

Thus $\boxed{\omega = \frac{2\pi}{T} = 2\pi f.}$ (4.2)

Velocity

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}[x_m \cos(\omega t + \phi)], \quad (4.3a)$$

$$v(t) = -\omega x_m \sin(\omega t + \phi)]. \quad (4.3b)$$

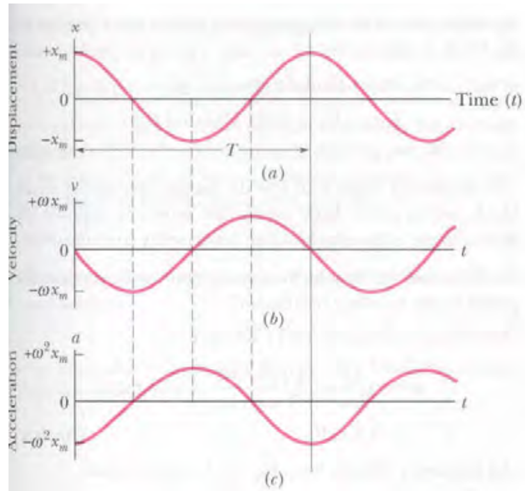
Velocity amplitude: $v_m = \omega x_m$ (4.4).

Acceleration

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}[-\omega x_m \sin(\omega t + \phi)],$$

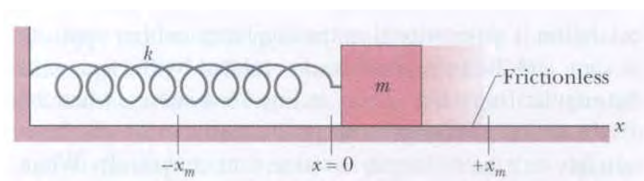
$$a(t) = -\omega^2 x_m \cos(\omega t + \phi)]. \quad (4.5)$$

Acceleration amplitude $a_m = \omega^2 x_m$ (4.6).



This equation of motion will be very useful in identifying simple harmonic motion and its frequency.

4.2 The Force Law for Simple Harmonic Motion



Consider the simple harmonic motion of a block of mass *m* subject to the elastic force of a spring

$$F = -kx \text{ (Hook's Law)} \quad (4.7).$$

Newton's law:

$$F = -kx = ma.$$

$$m\frac{d^2x}{dt^2} + kx = 0.$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \quad (4.8)$$

Comparing with the equation of motion for simple harmonic motion,

$$\omega^2 = \frac{k}{m}. \tag{4.9}$$

Simple harmonic motion is the motion executed by an object of mass *m* subject to a force that is proportional to the displacement of the object but opposite in sign.

Angular frequency:

$$\omega = \sqrt{\frac{k}{m}} \tag{4.10}$$

Period:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{4.11}$$

Examples 4.1

A block whose mass *m* is 680 g is fastened to a spring whose spring constant *k* is 65 Nm^{-1}. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(a) What force does the spring exert on the block just before the block is released?
(b) What are the angular frequency, the frequency, and the period of the resulting oscillation?
(c) What is the amplitude of the oscillation?
(d) What is the maximum speed of the oscillating block?
(e) What is the magnitude of the maximum acceleration of the block?
(f) What is the phase constant ϕ for the motion?

Answers:

(a) $F = -kx = -65 \times 0.11 = -7.2\text{ N}$

(b) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65}{0.68}} = 9.78\text{ rad s}^{-1}$

$f = \frac{\omega}{2\pi} = 1.56\text{ Hz}$

$T = \frac{1}{f} = 0.643\text{ s}$

(c) $x_m = 11\text{ cm}$

(d) $v_m = \omega x_m = 1.08\text{ ms}^{-1}$

(e) $a_m = \omega^2 x_m = 9.78^2 \times 0.11 = 10.5\text{ ms}^{-2}$

(f) At $t = 0$, $x(0) = x_m \cos\phi = 0.11$ (1)

$v(0) = -\omega x_m \sin\phi = 0$ (2)

(2): $\sin\phi = 0 \Rightarrow \phi = 0$

Example 4.2

At t = 0, the displacement of $x(0)$ of the block in a linear oscillator is – 8.50 cm. Its velocity $v(0)$ then is – 0.920 ms^{-1}, and its acceleration $a(0)$ is 47.0 ms^{-2}.

(a) What are the angular frequency ω and the frequency f of this system?
(b) What is the phase constant ϕ?
(c) What is the amplitude x_m of the motion?

(a) At $t = 0$,

$$x(t) = x_m \cos(\omega t + \phi) = x_m \cos\phi = -0.085. \quad (1)$$

$$v(t) = -\omega\, x_m \sin(\omega t + \phi) = -\omega\, x_m \sin\phi = -0.920. \quad (2)$$

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi) = -\omega^2 x_m \cos\phi = +47.0. \quad (3)$$

(3) ÷ (1): $\frac{a(0)}{x(0)} = -\omega^2.$

$$\omega = \sqrt{-\frac{a(0)}{x(0)}} = \sqrt{-\frac{47.0}{-0.0850}} = 23.5 \text{ rad s}^{-1}. \text{ (answer)}$$

(b) (2) ÷ (1): $\frac{v(0)}{x(0)} = -\omega \frac{\sin\phi}{\cos\phi} = -\omega \tan\phi.$

$$\tan\phi = -\frac{v(0)}{\omega x(0)} = -\frac{-0.920}{(23.51)(-0.085)} = -0.4603.$$

$\phi = -24.7^o$ or $180^o - 24.7^o = 155^o$.

One of these 2 answers will be chosen in (c).

(c) (1): $x_m = \frac{x(0)}{\cos\phi}.$

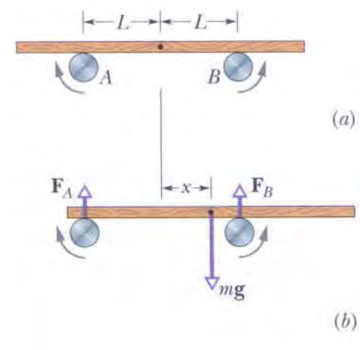
For $\phi = -24.7^o$, $x_m = \frac{-0.085}{\cos(-24.7^o)}$ m $= -9.4$ cm.

For $\phi = 155^o$, $x_m = \frac{-0.085}{\cos 155^o}$ m $= 9.4$ cm.

Since x_m is positive, $\phi = 155^o$ and $x_m = 9.4$ cm.
(answer)

Example 4.3

A uniform bar with mass m lies symmetrically across two rapidly rotating, fixed rollers, A and B, with distance L = 2.0 cm between the bar's centre of mass and each roller. The rollers slip against the bar with coefficient of kinetic friction μ_k = 0.40. Suppose the bar is displaced horizontally by a distance x, and then released. What is the angular frequency ω of the resulting horizontal simple harmonic (back and forth) motion of the bar?



Newton's law:

$\sum F_y = F_A + F_B - mg = 0.$ (1)

$\sum F_x = f_{kA} + f_{kB} = ma.$ (2)

or $\mu_k F_A - \mu_k F_B = ma.$

Considering torques about A,

$\sum \tau_z = F_A \cdot 0 + F_B \cdot 2L - mg(L+x) + f_{kA} \cdot 0 + f_{kB} \cdot 0 = 0.$ (3)

or $F_B \cdot 2L = mg(L+x).$

(3): $F_B = \dfrac{mg(L+x)}{2L}.$

(1): $F_A = mg - F_B = \dfrac{mg(L-x)}{2L}.$

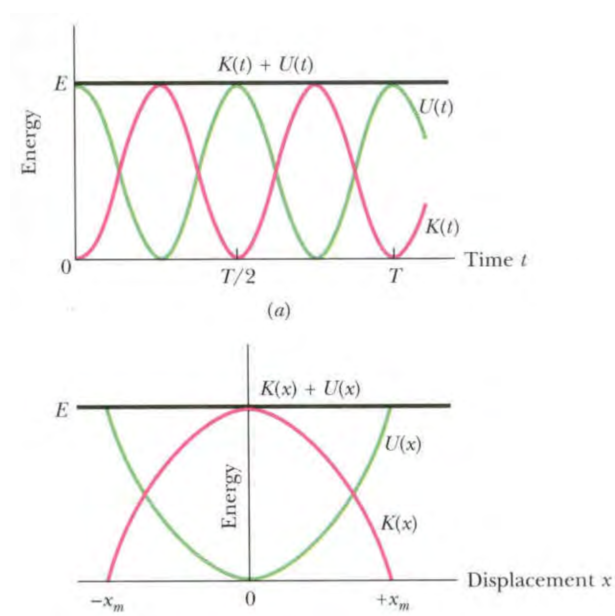
(2): $\dfrac{\mu_k}{2L}[mg(L+x) - mg(L-x)] = ma.$

$$\frac{d^2x}{dt^2} + \frac{g\mu_k}{L}x = 0.$$

Comparing with $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$ for simple harmonic motion, $\omega^2 = \dfrac{\mu_k g}{L}$,

$\omega = \sqrt{\dfrac{\mu_k g}{L}} = \sqrt{\dfrac{(0.40)(9.8)}{0.02}} = 14 \text{ rad s}^{-1}.$ (answer)

4.3 Energy in Simple Harmonic Motion



Potential energy:

Since $x(t) = x_m \cos(\omega t + \phi)$,

$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi).$ (4.12)

Kinetic energy:

Since $v(t) = -\omega\, x_m \sin(\omega t + \phi)$,

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi). \qquad (4.13)$$

Since $\omega^2 = k/m$,

$$K(t) = \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi).$$

Mechanical energy:

$$E = U + K = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi) + \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi)$$
$$= \frac{1}{2}kx_m^2 [\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)].$$

Since $\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi) = 1$,

$$\boxed{E = U + K = \frac{1}{2}kx_m^2.} \qquad (4.14)$$

(a) The potential energy $U(t)$, kinetic energy $K(t)$, and mechanical energy E as functions of time, for a linear harmonic oscillator. Note that all energies are positive and that the potential energy and kinetic energy peak twice during every period. (b) The potential energy $U(t)$, kinetic energy $K(t)$, and mechanical energy E as functions of position, for a linear harmonic oscillator with amplitude X_m. For $x = 0$ the energy is all kinetic, and for $x = \pm X_m$ it is all potential.

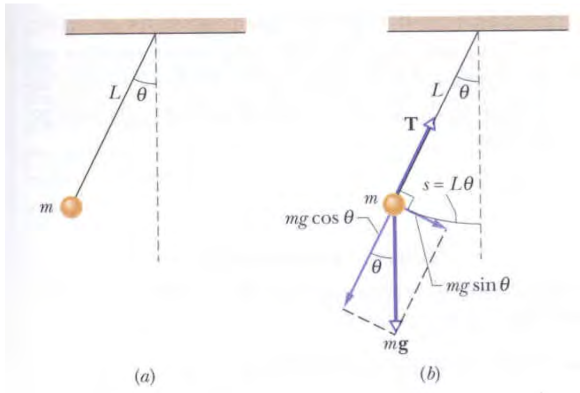
The mechanical energy is conserved.

4.4 The Simple Pendulum

Consider the tangential motion acting on the mass. Using Newton's law of motion,

$$-mg\sin\theta = ma = mL\frac{d^2\theta}{dt^2},$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0. \qquad (4.14)$$



When the pendulum swings through a small angle, $\sin\theta \approx \theta$. Therefore

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \qquad (4.15)$$

Comparing with the equation of motion for simple harmonic motion, $\omega^2 = \dfrac{g}{L}$ and $T = 2\pi\sqrt{\dfrac{L}{g}}$.

5 The Centre of Mass

5.1 Definition

The centre of mass of a body or a system of bodies is *the point* that moves as though all of the mass were concentrated there and all external forces were applied there.

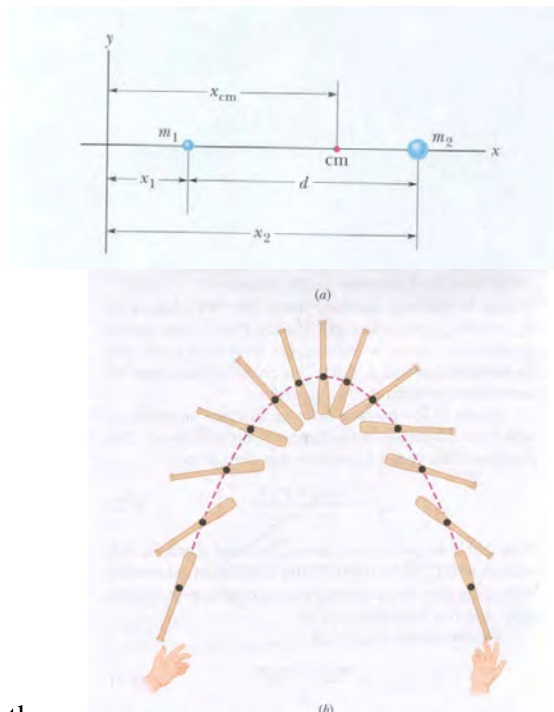


FIGURE 9-1 (*a*) A ball tossed into the air follows a parabolic path. (*b*) The center of mass (the black dot) of a baseball bat that is flipped into the air does also, but all other points of the bat follow more complicated curved paths.

For two particles,

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}.$$

For n particles,

$$x_{cm} = \frac{m_1 x_1 + \cdots + m_n x_n}{M}.$$

In general and in vector form,

$$\boxed{\vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i.} \qquad (5.1)$$

If the particles are in a uniform gravity field, then the total torque relative to the center of mass is zero. The same applies for the inertia force when the particles are in an accelerating reference frame.

Proof:

$$\vec{\tau} = \sum_{i=1}^{n} m_i(\vec{r}_i - \vec{r}_{cm}) \times \vec{g} = (\sum_{i=1}^{n} m_i \vec{r}_i - \vec{r}_{cm}\sum_{i=1}^{n} m_i) \times \vec{g} = (M\vec{r}_{cm} - M\vec{r}_{cm}) \times \vec{g} = 0$$

5.2 Rigid Bodies

$$x_{cm} = \frac{1}{M}\int x dm = \frac{1}{M}\int x\rho \, dV,$$
$$y_{cm} = \frac{1}{M}\int y dm = \frac{1}{M}\int y\rho \, dV, \qquad (5.2)$$
$$z_{cm} = \frac{1}{M}\int z dm = \frac{1}{M}\int z\rho \, dV$$

where ρ is the mass density.

If the object has uniform density,

$$\rho = \frac{dm}{dV} = \frac{M}{V}. \qquad (5.3)$$

Rewriting $dm = \rho dV$ and $m = \rho V$, we obtain

$$x_{cm} = \frac{1}{V}\int x dV,$$
$$y_{cm} = \frac{1}{V}\int y dV, \qquad (5.4)$$
$$z_{cm} = \frac{1}{V}\int z dV.$$

Similar to a system of particles, if the rigid body is in a uniform gravity field, then the total torque relative to its center of mass is zero. This is true even when the density of the object is non-uniform. The same applies to the inertia force. The proof is very much the same as in the case for particles. One only needs to replace the summation by integration operations.

5.3 Newton's Second Law for a System of Particles

In terms of X-Y-Z components,

$$\sum F_{ext,x} = Ma_{cm,x},$$
$$\sum F_{ext,y} = Ma_{cm,y}, \qquad (5.5)$$
$$\sum F_{ext,z} = Ma_{cm,z}.$$

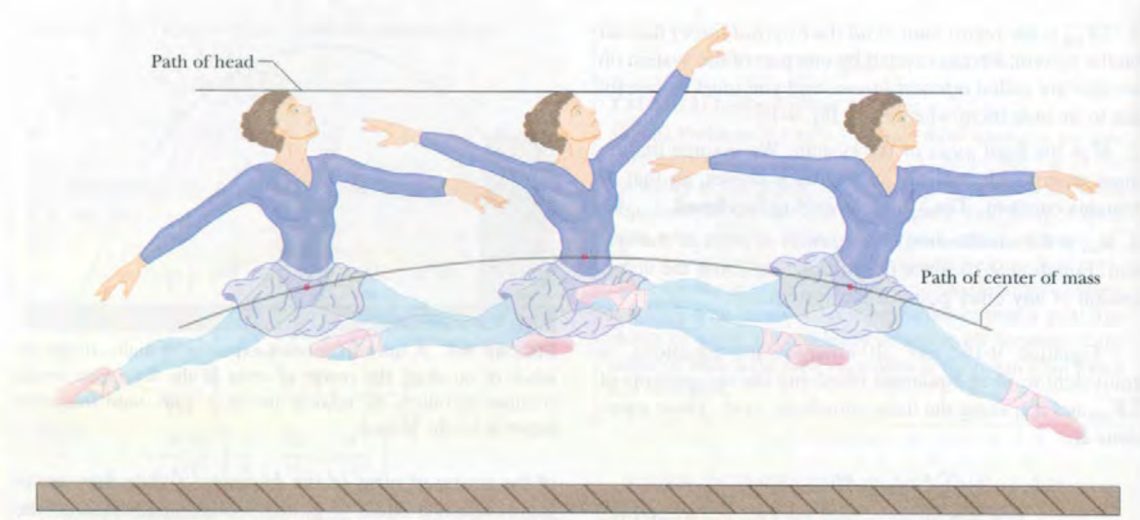


FIGURE 9-8 A grand jeté. (Adapted from *The Physics of Dance,* by Kenneth Laws, Schirmer Books, 1984.)

5.4 Linear Momentum

For a single particle, the linear momentum is

$$\boxed{\vec{p} = m\vec{v}.} \qquad (5.6)$$

Newton's Law:

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}. \qquad (5.7)$$

This is the most general form of Newton's Second Law. It accounts for the change of mass as well.

$$I = \int \sum \vec{F} dt = \vec{p}_f - \vec{p}_i. \qquad (5.8)$$

The change of momentum is equal to the time integration of the force, or *impulse*.

For a system of particles, the total linear momentum is

$$\vec{P} = \vec{p}_1 + \cdots + \vec{p}_n = m_1\vec{v}_1 + \cdots + m_n\vec{v}_n. \qquad (5.9)$$

Differentiating the position of the centre of mass,

$$M\vec{v}_{cm} = m_1\vec{v}_1 + \cdots + m_n\vec{v}_n.$$

$$\boxed{\vec{P} = M\vec{v}_{cm}.} \qquad (5.10)$$

The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the centre of mass.

Apply Newton's Law's to the particle system,

$$m_i \vec{a}_i(t) = \vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij},$$

$$\vec{X} = \frac{1}{M} \sum_i m_i \vec{x}_i, \qquad (M = \sum_i m_i = total \quad mass)$$

$$\Rightarrow \vec{V} = \frac{1}{M} \sum_i m_i \vec{v}_i$$

$$\Rightarrow \vec{A} = \frac{1}{M} \sum_i m_i \vec{a}_i = \frac{1}{M} \sum_i \left[\vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij} \right]$$

According to the Third Law, $\vec{F}_{ij} = -\vec{F}_{ji}$. So

$$\sum_{j \neq i} \vec{F}_{ij} = 0 \qquad (5.11)$$

and

$$\vec{A} = \frac{1}{M} \left[\sum_i \vec{F}_i^{ext} + (0) \right] = \frac{\vec{F}_{tot}^{ext}}{M} \qquad (5.12)$$

Newton's law:

$$\frac{d\vec{P}}{dt} = M \frac{d\vec{v}_{cm}}{dt} = M\vec{a}_{cm}. \qquad (5.13)$$

Hence

$$\boxed{\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}.} \qquad (5.14)$$

If $\sum \vec{F}_{ext} = 0$, then $M\vec{V} = \sum_i m_i \vec{v}_i = const$ (5.15)

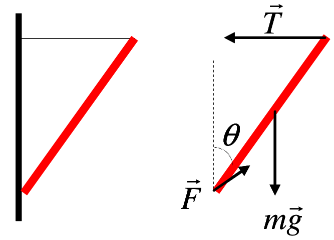
The total momentum of a system is conserved if the total external force is zero.

5.5 Rigid body at rest

The necessary and sufficient conditions for the balance of a rigid body is that net external force = 0, and the net torque due to these external forces = 0, relative to *any* origin (pivot). Choosing an appropriate origin can sometimes greatly simplify the problems. A common trick is choosing the origin at the point where an unknown external force is acting upon.

Example 5.1

A uniform rod of length $2l$ and mass m is fixed on one end by a thin and horizontal rope, and on the wall on the other end. Find the tension in the rope and the force of wall acting upon the lower end of the rod.



Answer:

The force diagram is shown. Choose the lower end as the origin, so the torque of the unknown force $\vec{F}$ is zero. By balance of the torque due to gravity and the tension, we get $mgl\sin\theta$ - $Tl\cos\theta = 0$, or $T = mg\tan\theta$

Breaking $\vec{F}$ along the X-Y (horizontal-vertical) directions, we get $F_x = T$, and $F_y = mg$.
It is interesting to explore further. Let us choose another point of origin for the consideration of torque balance. One can easily verify that with the above answers the total torque is balanced relative to any point of origin, like the center of the rod, or the upper end of the rod.

Can you prove the following?
If a rigid body is at rest, the total torque relative to any pivot point is zero.

5.6 Conservation of Linear Momentum

If the system of particles is isolated (i.e. there are no external forces) and closed (i.e. no particles leave or enter the system), then

$$\boxed{\vec{P} = \text{constant.}} \qquad (5.16)$$

Law of conservation of linear momentum:

$$\vec{P}_i = \vec{P}_f. \qquad (5.17)$$

Example 5.2

Imagine a spaceship and cargo module, of total mass M, traveling in deep space with velocity v_i = 2100 km/h relative to the Sun. With a small explosion, the ship ejects the cargo module, of mass $0.20M$. The ship then travels 500 km/h faster than the module; that is, the relative speed v_{rel} between the module and the ship is 500 km/h. What then is the velocity v_f of the ship relative to the Sun?

Using conservation of linear momentum,

$P_i = P_f$

$MV_i = 0.2M(v_f - v_{rel}) + 0.8Mv_f$

$v_i = v_f - 0.2v_{rel}$

$v_f = v_i + 0.2v_{rel}$

= 2100 + (0.2)(500)

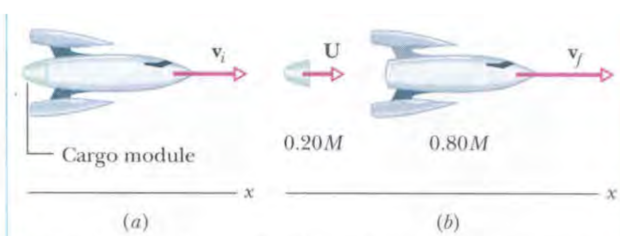


FIGURE 9-12 Sample Problem 9-9. (a) A spaceship, with a cargo module, moving at velocity $\mathbf{v}_i$. (b) The spaceship has ejected the cargo module. The spaceship now moves at velocity $\mathbf{v}_f$ and the cargo module moves at velocity **U**.

= 2200 km/h (answer)

Example 5.3

Two blocks are connected by an ideal spring and are free to slide on a frictionless horizontal surface. Block 1 has mass m_1 and block 2 has mass m_2. The blocks are pulled in opposite directions (stretching the spring) and then released from rest.

(a) What is the ratio v_1/v_2 of the velocity of block 1 to the velocity of block 2 as the separation between the blocks decreases?
(b) What is the ratio K_1/K_2 of the kinetic energies of the blocks as their separation decreases?

Answer
(a) Using conservation of linear momentum,

$$P_i = P_f$$

$$0 = m_1 v_1 + m_2 v_2$$

$$\frac{v_1}{v_2} = -\frac{m_2}{m_1}$$

(b) $$\frac{K_1}{K_2} = \frac{\frac{1}{2}m_1 v_1^2}{\frac{1}{2}m_2 v_2^2} = \frac{m_1}{m_2}\left(\frac{v_1}{v_2}\right)^2 = \frac{m_1}{m_2}\left(-\frac{m_2}{m_1}\right)^2 = \frac{m_2}{m_1}$$

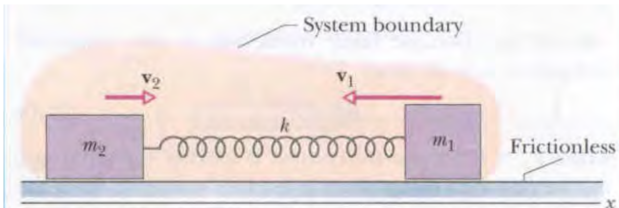


FIGURE 9-13 Sample Problem 9-10. Two blocks, resting on a frictionless surface and connected by a spring, have been pulled apart and then released from rest. The vector sum of their linear momenta remains zero during their subsequent motions. The system boundary is shown.

Example 5.4

A firecracker placed inside a coconut of mass M, initially at rest on a frictionless floor, blows the fruit into three pieces and sends them sliding across the floor. An overhead view is shown in the figure. Piece C, with mass $0.30M$, has final speed v_{fc}=5.0ms^{-1}.

(a) What is the speed of piece B, with mass $0.20M$?
(b) What is the speed of piece A?

Answer:
(a) Using conservation of linear momentum,
(b) $P_{ix} = P_{fx}$

$$P_{iy} = P_{fy}$$

$$m_C v_{fC} \cos 80^{\circ} + m_B v_{fB} \cos 50^{\circ} - m_A v_{fA} = 0 \qquad (1)$$

$$m_C v_{fC} \sin 80^{\circ} - m_B v_{fB} \sin 50^{\circ} = 0 \qquad (2)$$

$m_A = 0.5M$, $m_B = 0.2M$, $m_C = 0.3M$.

(2): $0.3Mv_{fC} \sin 80^{\circ} - 0.2Mv_{fB} \sin 50^{\circ} = 0$

$$v_{fB} = \frac{(0.3)(5)\sin 80^{\circ}}{0.2\sin 50^{\circ}} = 9.64 \text{ ms}^{-1} \approx 9.6 \text{ ms}^{-1} \text{ (answer)}$$

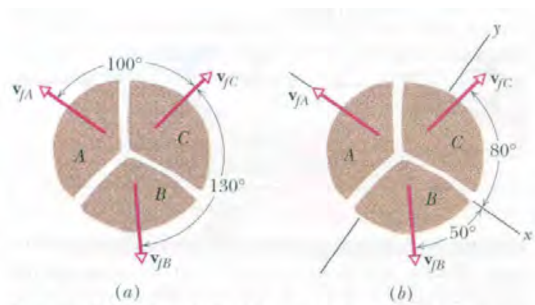


FIGURE 9-14 Sample Problem 9-11. Three pieces of an exploded coconut move off in three directions along a frictionless floor. (*a*) An overhead view of the event. (*b*) The same with a two-dimensional axis system imposed.

(b) (1): $0.3Mv_{fC}\cos 80^{\circ} + 0.2Mv_{fB}\cos 50^{\circ} = 0.5Mv_{fA}$

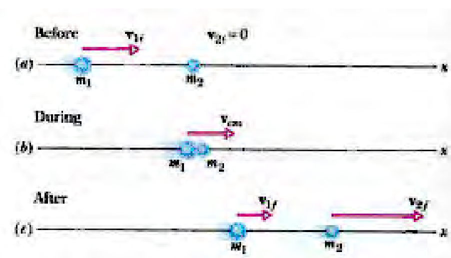
$$v_{fA} = \frac{(0.3)(5)\cos 80^{\circ} + (0.2)(9.64)\cos 50^{\circ}}{0.5} = 3.0\ \text{ms}^{-1}$$

(answer)

5.7 Elastic Collisions in One Dimension

In an elastic collision, the kinetic energy of each colliding body can change, but the total kinetic energy of the system does not change.

In a closed, isolated system, the linear momentum of each colliding body can change, but the net linear momentum cannot change, regardless of whether the collision is elastic.



In the case of stationary target, conservation of linear momentum:

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}.$$

Conservation of kinetic energy:

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2.$$

Rewriting these equations as

$$\boxed{m_1(v_{1i} - v_{1f}) = m_2 v_{2f},}$$

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2 v_{2f}^2.$$

Dividing,

$$\boxed{v_{1i} + v_{1f} = v_{2f}.}$$

We have two linear equations for v_{1f} and v_{2f}. Solutic

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i},$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}.$$

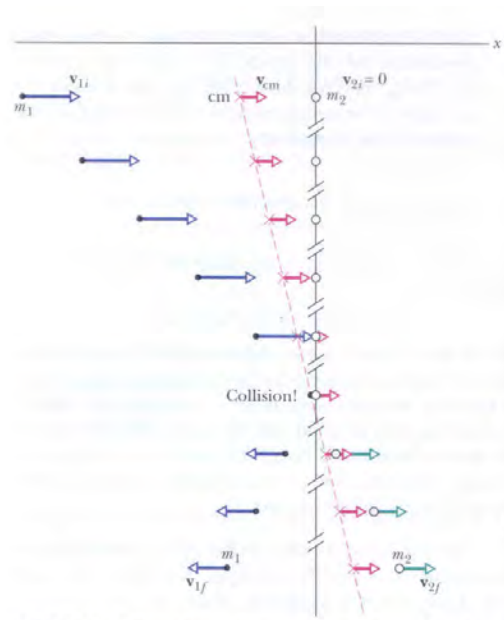


FIGURE 10-8 Some freeze-frames of two bodies undergoing an elastic collision. Body 2 is initially at rest, and $m_2 = 3m_1$. The velocity of the center of mass is also shown. Note that it is unaffected by the collision.

Motion of the centre of mass: $$v_{cm} = \frac{P}{m_1 + m_2} = \frac{m_1}{m_1 + m_2} v_{1i}.$$

Example 5.5

In a nuclear reactor, newly produced fast neutrons must be slowed down before they can participate effectively in the chain-reaction process. This is done by allowing them to collide with the nuclei of atoms in a *moderator*.

(a) By what fraction is the kinetic energy of a neutron (of mass m_1) reduced in a head-on elastic collision with a nucleus of mass m_2, initially at rest?

(b) Evaluate the fraction for lead, carbon, and hydrogen. The ratios of the mass of a nucleus to the mass of a neutron (= m_2/m_1) for these nuclei are 206 for lead, 12 for carbon and about 1 for hydrogen.

Answer

(a) Conservation of momentum

$$m_1 v_{1i} = m_1 v_{if} + m_2 v_{2f}$$

For elastic collisions,

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

$$m_1 (v_{1i} - v_{1f}) = m_2 v_{2f} \quad (1)$$

$$m_1 (v_{1i}^2 - v_{1f}^2) = m_2 v_{2f}^2 \quad (2)$$

Dividing (1) over (2), $v_{1i} + v_{1f} = v_{2f}$ (3)

(1): $m_1 (v_{1i} - v_{1f}) = m_2 (v_{1i} + v_{1f})$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

Fraction of kinetic energy reduction

$$= \frac{K_i - K_f}{K_i} = \frac{\frac{1}{2}m_1 v_{1i}^2 - \frac{1}{2}m_1 v_{1f}^2}{\frac{1}{2}m_1 v_{1i}^2} = \frac{v_{1i}^2 - v_{1f}^2}{v_{1i}^2} = 1 - \frac{v_{1f}^2}{v_{1i}^2}$$

$$= 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 = \frac{4m_1 m_2}{(m_1 + m_2)^2} \text{ (answer)}$$

(b) For lead, $m_2 = 206m_1$,

$$\text{Fraction } = \frac{4m_1(206m_1)}{(m_1 + 206m_1)^2} = \frac{4(206)}{207^2} = 1.9\% \text{ (answer)}$$

For carbon, $m_2 = 12m_1$,

$$\text{Fraction } = \frac{4m_1(12m_1)}{(m_1 + 12m_1)^2} = \frac{4(12)}{13^2} = 28\% \text{ (answer)}$$

For hydrogen, $m_2 = m_1$,

$$\text{Fraction } = \frac{4m_1(m_1)}{(m_1 + m_1)^2} = 100\% \text{ (answer)}$$

In practice, water is preferred.

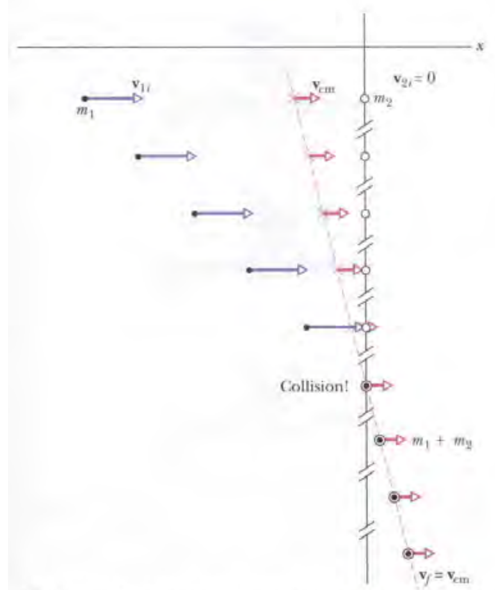


FIGURE 10-12 Some freeze-frames of two bodies undergoing a completely inelastic collision. Body 2 is initially at rest, and $m_2 = 3m_1$. The bodies stick together after the collision and move forward together. The velocity of the center of mass is also shown. Note that it is unaffected by the collision and that it is equal to the final velocity of the stuck-together bodies.

5.8 Inelastic Collisions in One Dimension

In an inelastic collision, the kinetic energy of the system of colliding bodies is not conserved.

In a completely inelastic collision, the colliding bodies stick together after the collision.

However, *the conservation of linear momentum still holds*.

$$m_1 v = (m_1 + m_2)V, \text{ or } V = \frac{m_1}{m_1 + m_2} v.$$

Examples 5.6

The *ballistic pendulum* was used to measure the speeds of bullets before electronic timing devices were developed. Here it consists of a large block of wood of mass M = 5.4 kg, hanging from two long cords. A bullet of mass m = 9.5 g is fired into the block, coming quickly to rest. The *block* + *bullet* then swing upward, their centre of mass rising a vertical distance h = 6.3 cm before the pendulum comes momentarily to rest at the end of its arc.

(a) What was the speed v of the bullet just prior to the collision?

(b) What is the initial kinetic energy of the bullet? How much of this energy remains as mechanical energy of the swinging pendulum?

Answer

(a) Using conservation of momentum during collision,

$$mv = (M + m)V$$

Using conservation of energy after collision,

$$\frac{1}{2}(M + m)V^2 = (M + m)gh$$

$$V = \sqrt{2gh}$$

$$v = \frac{M + m}{m}V$$

$$= \frac{M + m}{m}\sqrt{2gh} = \frac{5.4 + 0.0095}{0.0095}\sqrt{2(9.8)(0.063)} = 630\ \text{ms}^{-1}$$

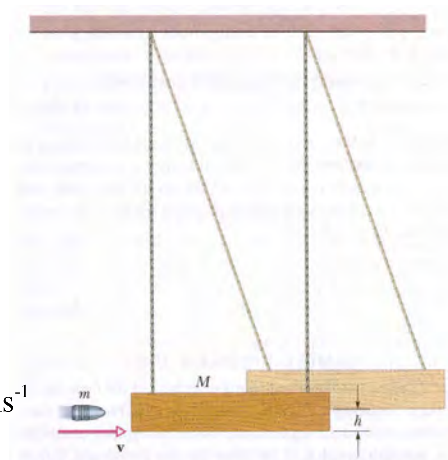


FIGURE 10-14 Sample Problem 10-5. A ballistic pendulum, used to measure the speeds of bullets.

(b) Initial kinetic energy

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(0.0095)630^2 = 1900\ \text{J}$$

Final mechanical energy

$$E = (M + m)gh = (5.4 + 0.0095)(9.8)(0.063) = 3.3\ \text{J}$$

(only 0.2%) (answer)

Example 5.7

(The Physics of Karate) A karate expert strikes downward with his fist (of mass m_1 = 0.70 kg), breaking a 0.14 kg wooden board. He then does the same to a 3.2 kg concrete block. The spring constants k for bending are $4.1 \times 10^4\ \text{Nm}^{-1}$ for the board and $2.6 \times 10^6\ \text{Nm}^{-1}$ for the block. Breaking occurs at a deflection d of 16 mm for the board and 1.1 mm for the block.

(a) Just before the board and block break, what is the energy stored in each?

(b) What fist speed v is required to break the board and the block? Assume that mechanical energy is conserved during the bending, that the fist and struck object stop just before the break, and that the fist-object collision at the onset of bending is completely inelastic.

Answer

(a) For the board, $U = \frac{1}{2}kd^2 = \frac{1}{2}(4.1\times10^4)0.016^2 = 5.248\ \text{J} \approx 5.2\ \text{J}$ (answer)

For the block, $U = \frac{1}{2}kd^2 = \frac{1}{2}(2.6\times10^6)0.0011^2 = 1.573\text{ J}$ ≈ 1.6 J (answer)

(b) For the board, first the fist and the board undergoes an inelastic collision. Conservation of momentum:

$$m_1 v = (m_1 + m_2)V \qquad (1)$$

Then the kinetic energy of the fist and the board is converted to the bending energy of the wooden board. Conservation of energy:

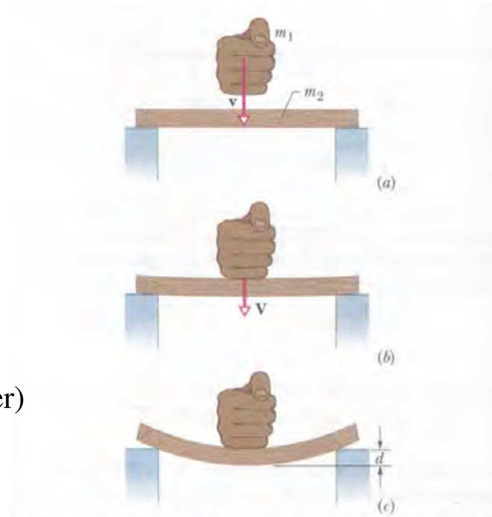


FIGURE 10-15 Sample Problem 10-6. (*a*) A karate expert strikes at a flat object with speed *v*. (*b*) Fist and object undergo a completely inelastic collision, and bending begins. The *fist + object* then have speed *V*. (*c*) The object breaks when its center has been deflected by an amount *d*.

$$(2):\ V = \sqrt{\frac{2U}{m_1+m_2}} = \sqrt{\frac{2(5.248)}{0.7+0.14}} = 3.534\text{ ms}^{-1}$$

$$(1):\ v = \frac{m_1+m_2}{m_1}V = \left(\frac{0.7+0.14}{0.7}\right)3.534 \approx 4.2\text{ ms}^{-1}\text{ (answer)}$$

For the concrete block,

$$V = \sqrt{\frac{2U}{m_1+m_2}} = \sqrt{\frac{2(1.573)}{0.7+3.2}} = 0.8981\text{ ms}^{-1}.$$

$$v = \frac{m_1+m_2}{m_1}V = \left(\frac{0.7+3.2}{0.7}\right)0.8981 \approx 5.0\text{ ms}^{-1}\text{ (answer)}$$

The energy to break the concrete block is 1/3 of that for the wooden board, but the fist speed required to break the concrete block is 20% faster! This is because the larger mass of the block makes the transfer of energy to the block more difficult.

5.9 Collisions in Two Dimensions

Conservation of linear momentum:

x component: $m_1 v_{1i} = m_1 v_{1f}\cos\theta_1 + m_2 v_{2f}\cos\theta_2$,

y component: $0 = -m_1 v_{1f}\sin\theta_1 + m_2 v_{2f}\sin\theta_2$.

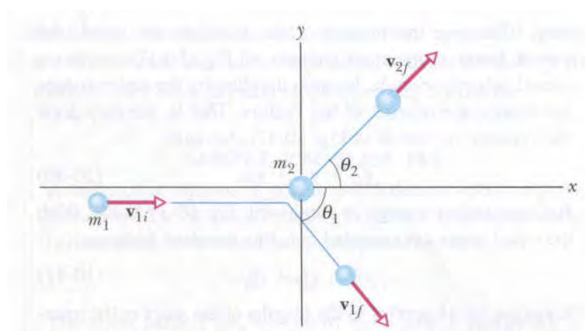


FIGURE 10-16 An elastic collision between two bodies in which the collision is not head-on. The body with mass m_2 (the target) is initially at rest.

Conservation of kinetic energy:

$$\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2.$$

Typically, we know m_1, m_2, v_{1i} and θ_1. Then we can solve for v_{1f}, v_{2f} and θ_2.

Examples 5.8

Two particles of equal masses have an elastic collision, the target particle being initially at rest. Show that (unless the collision is head-on) the two particles will always move off perpendicular to each other after the collision.

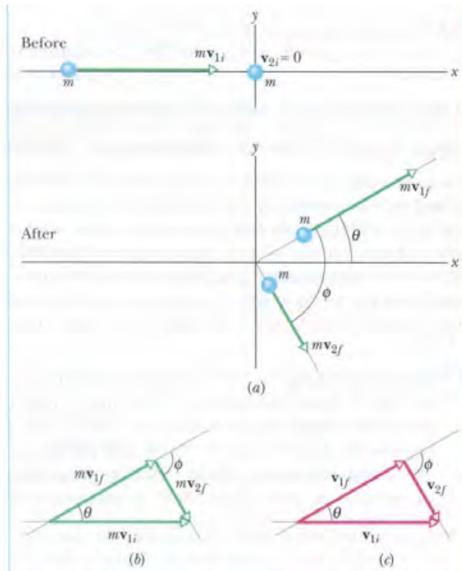


FIGURE 10-17 Sample Problem 10-7. A neat proof that in an elastic collision between two particles of the same mass, the particles fly off at 90° to each other afterward. For this to hold true, the target particle must be initially at rest and the collision must not be head-on.

Using conservation of momentum,

$$m\vec{v}_{1i} = m\vec{v}_{1f} + m\vec{v}_{2f}$$

$$\vec{v}_{1i} = \vec{v}_{1f} + \vec{v}_{2f}$$

The three vector form a triangle.
In this triangle, cosine law:

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f}\cos\phi \qquad (1)$$

Using conservation of energy:

$$\frac{1}{2}mv_{1i}^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2$$

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 \qquad (2)$$

(1) – (2): $2v_{1f}v_{2f}\cos\phi = 0$

$\phi = 90^\circ$ (answer)

Example 5.9

Two skaters collide and embrace, in a completely inelastic collision. That is, they stick together after impact. Alfred, whose mass m_A is 83 kg, is originally moving east with speed v_A = 6.2 km/h. Barbara, whose mass m_B is 55 kg, is originally moving north with speed v_B = 7.8 km/h.

(a) What is the velocity $\vec{V}$ of the couple after impact?
(b) What is the velocity of the centre of mass of the two skaters before and after the collision?
(c) What is the fractional change in the kinetic energy of the skaters because of the collision?

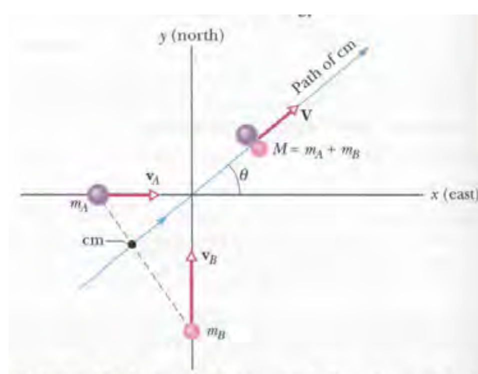


FIGURE 10-18 Sample Problem 10-8. Two skaters, Alfred (A) and Barbara (B), represented with spheres in this simplified overhead view, have a completely inelastic collision. Afterward, they move off together at angle θ, with speed V. The path of their center of mass is shown. The position of the center of mass for the indicated positions of the skaters before the collision is also shown.

(a) Conservation of momentum:

$$m_A v_A = (m_A + m_B)V\cos\theta \qquad (1)$$

$$m_B v_B = (m_A + m_B)V\sin\theta \qquad (2)$$

(2) ÷ (1):

$$\tan\theta = \frac{m_B v_B}{m_A v_A} = \frac{(55)(7.8)}{(83)(6.2)} = 0.834 \text{, so } \theta = 39.8^{\circ} \approx 40^{\circ} \text{ (answer)}$$

$$\text{(1): } V = \frac{m_A v_A}{(m_A + m_B)\cos\theta} = \frac{(83)(6.2)}{(83+55)\cos 39.8^{\circ}} = 4.86 \text{ km/h} \approx 4.9 \text{ km/h (answer)}$$

(b) Velocity of the centre of mass is not changed by the collision. Therefore $V = 4.9$ km/h and $\theta = 40^{\circ}$ both before and after the collision. (answer)

(c) Initial kinetic energy $K_i = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$ = 3270 kg km^2/h^2.

Final kinetic energy $K_f = \frac{1}{2}(m_A + m_B)V^2$ = 1630 kg km^2/h^2.

$$\text{Fraction } = \frac{K_i - K_f}{K_i} = \frac{1630 - 3270}{3270} = -50\% \text{ (answer)}$$

6 General equations of motion in the X-Y plane with constant forces

6.1 General formulae

$$\begin{aligned} v_x(t) &= v_{xo} + \left(\frac{F_x}{m}\right)t \\ v_y(t) &= v_{yo} + \left(\frac{F_y}{m}\right)t \end{aligned} \qquad (6.1)$$

$$\begin{aligned} x(t) &= x_o + v_{xo}t + \frac{1}{2}\left(\frac{F_x}{m}\right)t^2 \\ y(t) &= y_o + v_{yo}t + \frac{1}{2}\left(\frac{F_y}{m}\right)t^2 \end{aligned} \qquad (6.2)$$

These general formulae can be applied to any situations, once the initial conditions (x_0, y_0, v_{x0}, v_{y0}) are given.

A word of caution: Friction forces are not always 'constant'. Pay attention to their directions because they change with the direction of the velocity.

6.2 Projectile motion near Earth surface

The only force on the object is the gravity which is along the –Y direction. Accordingly, we have

$$\begin{aligned} v_x(t) &= v_{xo} + \left(\frac{F_x}{m}\right)t = v_{xo} \\ v_y(t) &= v_{yo} + \left(\frac{F_y}{m}\right)t = v_{yo} - gt \end{aligned} \qquad (6.3)$$

for the velocity and

$$x(t) = x_o + v_{xo}t + \frac{1}{2}\left(\frac{F_x}{m}\right)t^2 = x_o + v_{xo}t$$
$$y(t) = y_o + v_{yo}t + \frac{1}{2}\left(\frac{F_y}{m}\right)t^2 = y_o + v_{yo}t - \frac{1}{2}gt^2 \tag{6.4}$$

for the position.

These general formulae can be applied to any situations, once the initial conditions (x_0, y_0, v_{x0}, v_{y0}) are given.

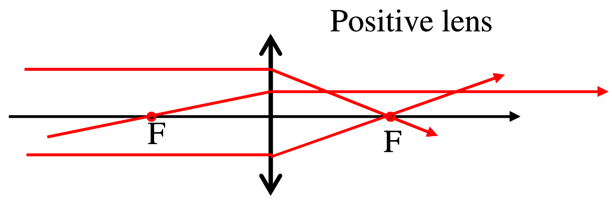
~end~

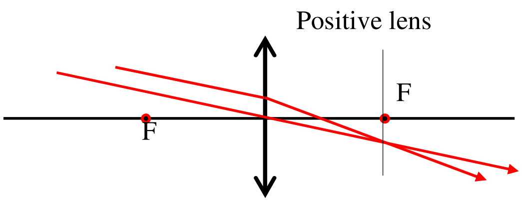
Geometric Optics

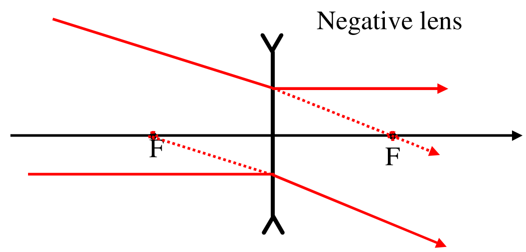
1 Lens

1.1 Basic properties

- Rays parallel to optical axis are focused to the focus point.
- Rays through the focus point become parallel to the optical axis
- Parallel rays are focused onto a point on the focal plane



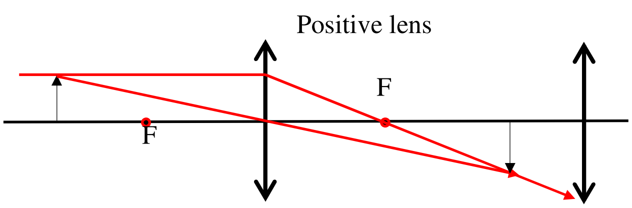


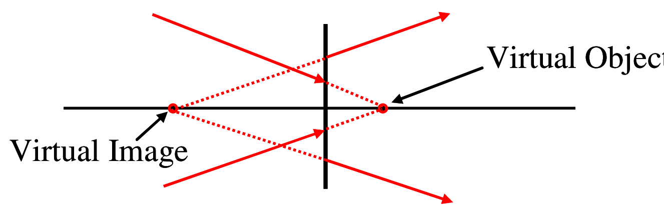


1.2 Finding the image of an object formed by a lens

s_o = object distance to lens, positive/negative if object on the left/right of lens
s_i = image distance to lens, positive/negative if object on the right/left of lens
f = focal length (property of individual lens, positive/negative for convex/concave lens)

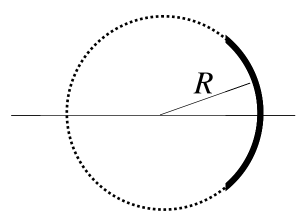
$$\frac{1}{s_o}+\frac{1}{s_i}=\frac{1}{f} \qquad (O.1)$$





1.3 Finding the image of an object formed by spherical mirror

s_o = object distance to lens, positive/negative if object on the left/right of mirror
s_i = image distance to lens, positive/negative if object on the left/right of mirror
f = focal length (= $R/2$, where R is the radius of the sphere, positive/negative for concave/convex mirror)



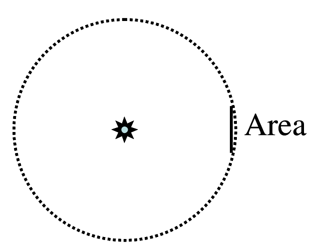
$$\frac{1}{s_o}+\frac{1}{s_i}=\frac{1}{f} \qquad (O.2)$$

1.4 Magnification

$$M = s_i/s_o \qquad (O.3)$$

1.5 Light intensity

Consider a point light source emitting light uniformly in all directions. The light energy through the sphere of radius r is therefore uniform, and the portion of energy



through a circular area of radius a is then $I = \frac{\pi a^2}{4\pi r^2} = \frac{1}{4}\left(\frac{a}{r}\right)^2$. For a lens of radius a at a distance d from a light source the energy flowing through it is $\propto \left(\frac{a}{r}\right)^2$.

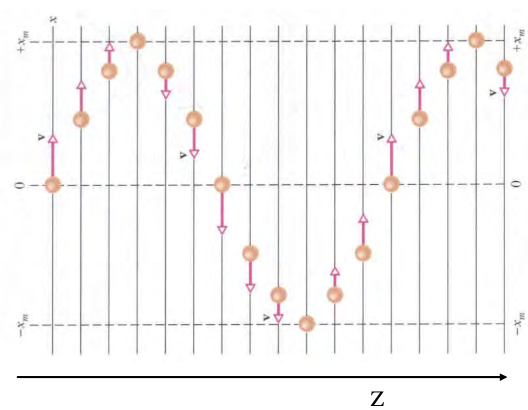
As shown, the brightness of the image depends on the size of the lens and the diameter of the aperture.

Waves

Definition: Collection of vibrating particles with fixed phase delay between neighbors

1 Wave equation

1.1 Plane wave



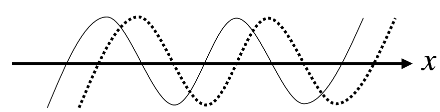
$$W(x,t) = A\cos(kx - \omega t) \qquad \text{(W.1)}$$

describes a plane wave propagating along the x-axis, i. e., the equal-phase surface is a plane, for example,

$$x = (\omega t - c)/k = \frac{\omega}{k}t - \frac{c}{k} \qquad \text{(W.2).}$$

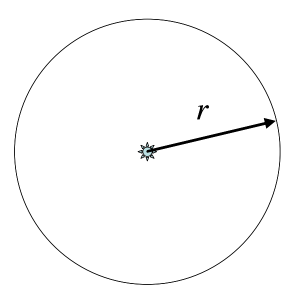
The plane is moving in space at a speed of $v = \frac{\omega}{k}$ (W.3

Here $k \equiv \frac{2\pi}{\lambda}$, and λ is the wavelength (W.4).



$W(x,t)$ is a periodic function of both time t and position x. The spatial period is λ the wavelength, i. e.,

$$W(x,t) = W(x+\lambda, t) \qquad \text{(W.5a),}$$



and the temporal period is $T \equiv \frac{2\pi}{\omega}$ (W.5b)

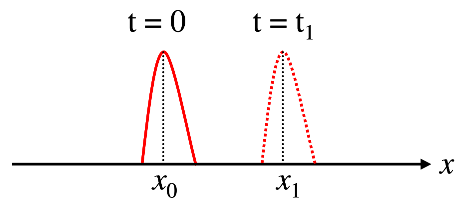
because $W(x,t) = W(x, t+T)$ (W.5c)

1.2 Spherical wave

$$W(r,t) = A\cos(kr - \omega t) \qquad \text{(W.6)}$$

i. e., the equal-phase surface is a sphere.

Pulse (shock) wave: $W(x,t) = f(x - vt) = f(x - x_0)$, with $x_0 = vt$, and v being the propagation speed. $f(x - x_0)$ is a function with maximum at $f(0)$.



1.3 Wave intensity

The intensity of a wave is the time average of the oscillation.

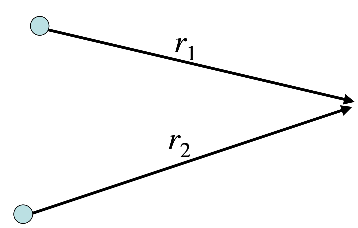
$$I \equiv \langle W(r,t)^2 \rangle = \frac{1}{T}\int_0^T W(r,t)^2 dt = \frac{1}{T}\int_0^T A^2 \cos^2(kr - \omega t) dt = \frac{1}{2}A^2 \qquad \text{(W.7)}$$

(Note that $\cos^2\theta = (1 - \cos 2\theta)/2$)

2 Wave interference

Waves from two sources meet.

$W_1 = A_1 \cos(kr_1 - \omega t + \phi_1)$, $W_2 = A_2 \cos(kr_2 - \omega t + \phi_2)$,



Superposition principle: $W = W_1 + W_2$ (W.8)

Using Eq. (W.7) we have $W^2 = W_1^2 + W_2^2 + 2W_1W_2$, so

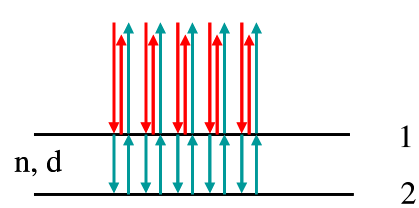
$$\langle W^2 \rangle = A_1^2 + A_2^2 + 2A_1A_2\cos\phi \qquad \text{(W.9)},$$

where the phase difference $\phi \equiv k(r_2 - r_1) + \phi_2 - \phi_1$ (W.9A).

Contrast $$I \equiv \frac{\langle W^2 \rangle_{\max} - \langle W^2 \rangle_{\min}}{\langle W^2 \rangle_{\max} + \langle W^2 \rangle_{\min}} = \frac{2A_1A_2}{A_1^2 + A_2^2} \qquad \text{(W.10)}.$$

I reaches maximum of '1' when $A_1 = A_2$.

The interference problems then become the problems of calculating path differences $\delta r \equiv r_2 - r_1$ and then the phase difference, which is usually straightforward to solve. For example, in the case of a thin film under normal incidence, the path difference between the waves reflected from the first surface (red arrows) and from the second surface (green arrows) is $2nd$, where n is the refractive index and d the thickness. The reflection at the first surface introduces a phase shift π, so the total phase difference is $\phi = \frac{2\pi nd}{\lambda} - \pi$.



Electrostatics

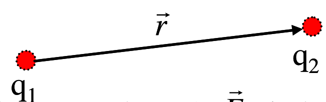
1 Electric charge

The crucial point is that some physical quantity which we call "charges" is discovered. The charges are associated with a special force we called electric force.

1.1 Coulomb's Law

This is the first Law of Physics on electromagnetism discovered by human beings. It says that the *electrostatic force* between two charges q_1 and q_2 separated by distance r is given by

$$\vec{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{r} \qquad \text{(E.1)}$$



where k is a constant, and $\hat{r}$ is a unit vector pointing in the direction from charge 1 to charge 2. $\vec{F}_{12}$ is the force acting on charge 2 from charge 1.

SI Unit of charge: Coulomb(C)

One coulomb is the amount of charge that is transferred through the cross section of a wire in 1 second when there is a current of 1 ampere in the wire.

Notice that the relationship between electric current and electric charges is already assumed in this definition. In SI Unit k is given by $k = \frac{1}{4\pi\varepsilon_0} = 8.99\times10^9\, N.m^2 / C^2$,

or $\varepsilon_0 = 8.85\times10^{-12} C^2 / N.m^2$.

For many charges, the forces satisfy the law of superposition,

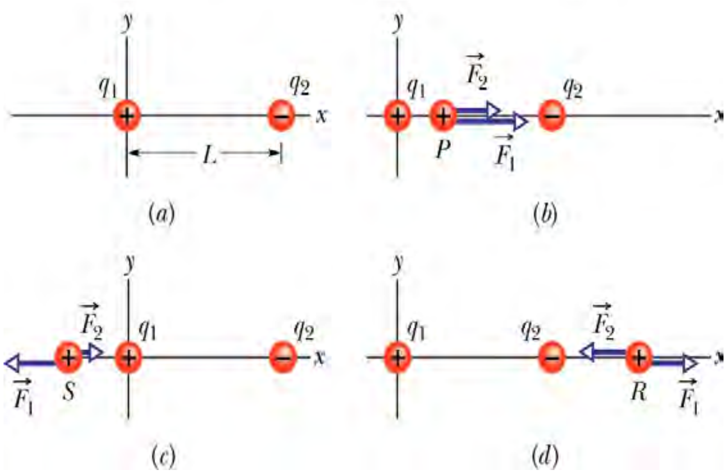
$$\vec{F}_{i,net} = \sum_{j\neq i}\vec{F}_{ij} = \frac{1}{4\pi\varepsilon_o}\sum_{j\neq 1}\frac{q_i q_j}{|\vec{r}_{ij}|^3}\vec{r}_{ij} \qquad \text{(E.2).}$$

Notice the similarity of Coulomb's Law with Law of Gravitation, $\vec{F}_{12} = G\frac{m_1 m_2}{r^2}\hat{r}$. The main difference is that charges can be both positive and negative, whereas masses are always positive. The similarity between Coulomb Law and Law of Gravitation also enable us to draw some conclusion about Coulomb forces easily. For example,

A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at its center.

If a charged particle is located inside a shell of uniform charge, there is no net electrostatic force on the particle from the shell.

Example:



The figure shows two particles fixed in place: a particle of charge $q_1 = 8q$ at the origin and a particle of charge $q_2 = -2q$ at $x = L$. At what point (other than infinitely far away) can a proton be placed so that it is in equilibrium? Is that equilibrium stable or unstable? ($-q$ = charge of electron)

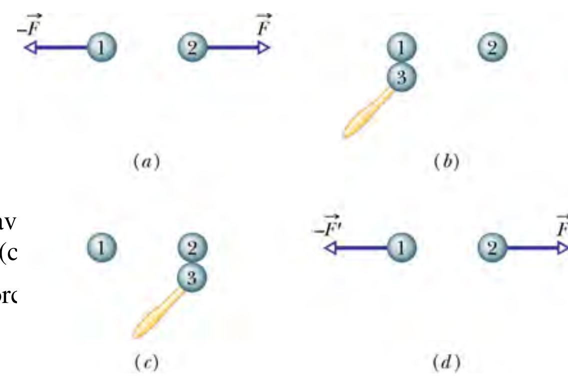
1.2 Spherical Conductors

If excess charges Q are placed on a piece of metal, the charge will move under each other's repulsion to stay as far away from each other as possible. That means they prefer to stay at the *surface of metal*. For spherical conductors with radius R the final distribution of charges is simple. Because of symmetry, the charges Q will be spread uniformly on the surface of the spherical conductor, the surface charge density being $\sigma = \frac{Q}{4\pi R^2}$.

1.3 Charge is quantized

We now know that like other materials in nature electric charges have a smallest unit, $e = 1.60\times10^{-19}C$, and all charges are multiple of the smallest unit, i.e. $q=ne$, $n = 0, \pm1, \pm2, \ldots$ etc. When a physical quantity such as charge can have only discrete values, we say that the quantity is *quantized*. It is in fact not obvious at all why matters in our universe all seem to be "quantized" somehow.

2) Identical isolated conducting spheres 1 and 2 have equal charges and are separated by a distance that is large compared with their diameters (fig.(a)). The electrostatic force acting on sphere 2 due to sphere 1 is $\vec{F}$.
Suppose now that a third identical sphere 3, hav … hed
first to sphere 1 (fig.(b)), then to sphere 2 (fig.(c … ıde
F, what is the magnitude of the electrostatic forc



2 Electric Fields

2.1 Introducing electric field

The concept of "Field" was initially introduced to describe forces between 2 objects that are separated by a distance (action at a distance). It is convenient to have a way of viewing how the force coming from object 1 felt by another object is "distributed" around object 1. This is particularly easy for charges obeying Coulomb's Law, since the force felt by charge 2 coming from object 1 is proportional to charge of object 2, i.e. we can write $\vec{F}_{12} = k\frac{q_1q_2}{r^2}\hat{r} = \vec{E}_{12}q_2$, where $\vec{E}_{12}$ is the *force per unit charge* acting on q_2 due to charge 1. Because of linearity of force, this concept can be generalized to the force felt by a test charge q at position $\vec{r}$ due to a distribution of other charges. In that case, we may write

$$\vec{F}(\vec{r}) = \sum_j \vec{F}_j = \frac{q}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j) = q\vec{E}(\vec{r}),$$

where $\vec{E}(\vec{r})$ is the force per unit charge of a charge q felt at position $\vec{r}$.

$$\vec{E}(\vec{r}) = \vec{F}(\vec{r})/q \qquad \text{(E.3)}$$

was given a name called *electric field*. The SI unit for electric field is obviously Newton per coulomb (N/C).

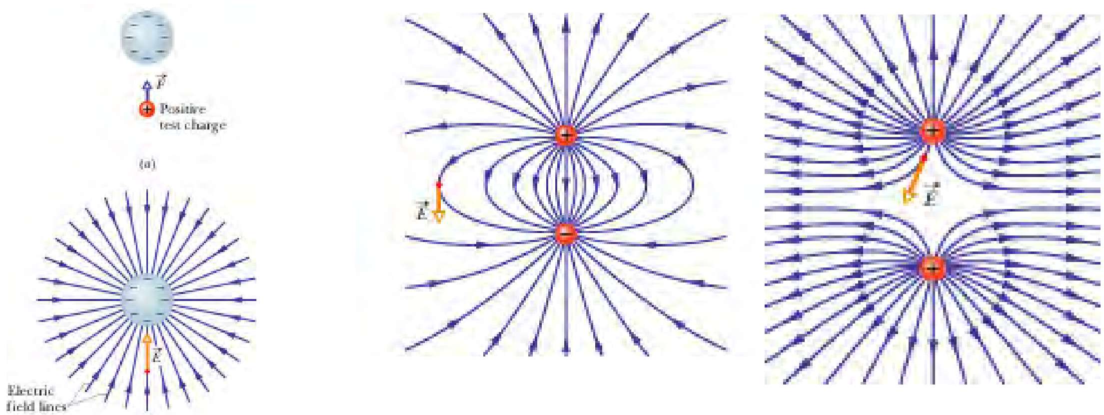
2.2 Electric Field Lines

To make it easier to visualize, Michael Faraday introduced the idea of *lines of force*, or electric field lines. Electric field lines are diagrams that represent electric fields. They are drawn with the following rules: (1) At any point, the direction of a straight field line or the direction of the tangent to a curved field line gives the direction of $\vec{E}$ at that point, and (2) the field lines are drawn so that the number of lines per unit area, measured in a plane that is perpendicular to the lines, is proportional to the magnitude of $\vec{E}$.

Some examples:

a) field lines from a (-) spherical charge distribution

b) Field lines from 2 point charges of equal magnitude (i) charges are same (ii) charges are opposite.



2.3 Electric field due to different charge distributions

1) Point charge at origin ($\vec{r} = 0$)

$$\vec{E} = \frac{q}{4\pi\varepsilon_o r^2}\hat{r} \qquad \text{(E.4)}$$

Exercise: what is the electric field at position $\vec{r} = (x, y, z)$ from 2 point charges with magnitude q, and q', respectively located at $\vec{r} = \pm z_0\hat{z}$?

We use the formula

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{(\vec{r}-z_0\hat{z})}{|\vec{r}-z_0\hat{z}|^3} + q'\frac{(\vec{r}+z_0\hat{z})}{|\vec{r}+z_0\hat{z}|^3}\right] \qquad \text{(E.5a)}$$

Therefore

$$E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x}{|\vec{r}-z_0\hat{z}|^3} + q'\frac{x}{|\vec{r}+z_0\hat{z}|^3}\right] \qquad \text{(E.5b),}$$

$$E_y(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{y}{|\vec{r}-z_0\hat{z}|^3} + q'\frac{y}{|\vec{r}+z_0\hat{z}|^3}\right] \quad \text{(E.5c)},$$

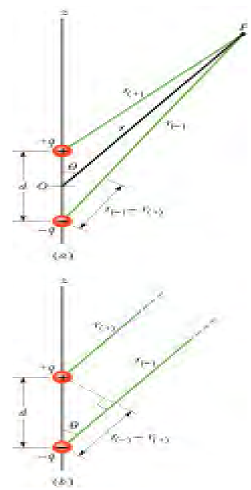
$$E_z(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{z-z_0}{|\vec{r}-z_0\hat{z}|^3} + q'\frac{z+z_0}{|\vec{r}+z_0\hat{z}|^3}\right] \quad \text{(E.5d)}$$

where $|\vec{r} \pm z_0\hat{z}| = \sqrt{x^2 + y^2 + (z \pm z_0)^2}$.

2) *Electric dipole*

This is the electric field due to 2 point charges with magnitude q , and $-q$, respectively located at $\vec{r}' = \pm\frac{d}{2}\hat{z}$, and at distances $|\vec{r}| >> d$ from the origin.

In math, $(1+x)^a \cong 1+ax$ when $x << 1$, and a is a real number.



Using the above result, we obtain for the dipole electric field;

$$E_x(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left[\frac{x}{|\vec{r}-\frac{d}{2}\hat{z}|^3} - \frac{x}{|\vec{r}+\frac{d}{2}\hat{z}|^3}\right] \underset{r>>d}{\sim} \frac{3qd}{4\pi\varepsilon_o}\frac{zx}{r^5} \quad \text{(E.6a)}$$

$$E_y(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left[\frac{y}{|\vec{r}-\frac{d}{2}\hat{z}|^3} - \frac{y}{|\vec{r}+\frac{d}{2}\hat{z}|^3}\right] \underset{r>>d}{\sim} \frac{3qd}{4\pi\varepsilon_o}\frac{zy}{r^5} \quad \text{(E.6b)}$$

$$E_z(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left[\frac{z-\frac{d}{2}}{|\vec{r}-\frac{d}{2}\hat{z}|^3} - \frac{z+\frac{d}{2}}{|\vec{r}+\frac{d}{2}\hat{z}|^3}\right] \underset{r>>d}{\sim} \frac{qd}{4\pi\varepsilon_o}\frac{3z^2-r^2}{r^5} \quad \text{(E.6c)}.$$

The electric field lines coming from a pair of opposite charges are shown in previous section (b). Notice that at distances far away from origin, the electric field is only proportional to the product *qd*. The quantity $\vec{p} = (qd)\hat{z}$ is called an electric dipole moment. The direction of the dipole is taken to be along the direction from the negative to the positive charge of a dipole.

2.4 Motion of Charges in electric field

a) *Point charge*

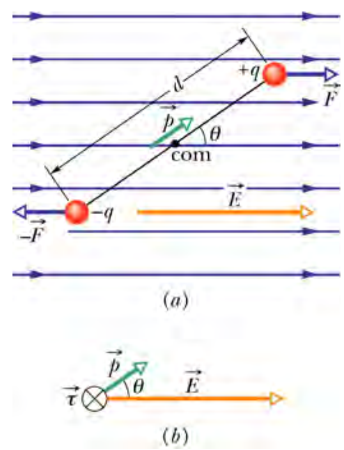
A particle with charge q satisfies Newton's Equation of motion $\vec{F} = m\vec{a}$ under electric field, where m is the mass of the particle. The force the particle feels is $\vec{F} = q\vec{E}$, followed from the definition of electric field.

b) *Dipole*

Since a dipole consists of 2 opposite charges of equal magnitude, the force acting on a dipole will be zero if electric is uniform. However, the forces on the charged ends do produce a net torque $\vec{\tau}$ on the dipole in general. The torque about its center of mass is

$$\tau = -(qE)(d/2)\sin\vartheta - (qE)(d/2)\sin\vartheta = -pE\sin\vartheta \qquad \text{(E.7)}$$

Notice that the torque is trying to rotate the dipole clockwise to decrease θ in the figure, which is why there is a (-) sign.



In vector form, $\vec{\tau} = \vec{p}\times\vec{E}$. Notice that associated with the torque is also a potential energy

$$U = -\int_{\pi/2}^{\vartheta} \tau d\vartheta = \int_{\pi/2}^{\vartheta} pE\sin\vartheta d\vartheta = -pE\cos\vartheta = -\vec{p}.\vec{E} \qquad \text{(E.8).}$$

c) *Cross-product of two vectors* $\vec{C} = \vec{A}\times\vec{B}$:

- $C = AB\sin\theta$, direction perpendicular to $\vec{A}$ and $\vec{B}$, and determined by right-hand-rule. (E.9a)
- $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ (E.9b)
- For unit vectors along the X-Y-Z axes

$$\vec{x}_0\times\vec{y}_0 = \vec{z}_0,\ \vec{y}_0\times\vec{z}_0 = \vec{x}_0,\ \vec{z}_0\times\vec{x}_0 = \vec{y}_0 \qquad \text{(E.10)}$$

$$\vec{A}\times\vec{B} = (A_x\vec{x}_0 + A_y\vec{y}_0 + A_z\vec{z}_0)\times(B_x\vec{x}_0 + B_y\vec{y}_0 + B_z\vec{z}_0)$$

$$= (A_yB_z - A_zB_y)\vec{x}_0 + (A_zB_x - A_xB_z)\vec{y}_0 + (A_xB_y - A_yB_x)\vec{z}_0 = \begin{vmatrix} \vec{x}_0 & \vec{y}_0 & \vec{z}_0 \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad \text{(E.11)}$$

Questions

What is the x-component of electric field at position $\vec{r} = (x, y, z)$ from 3 point charges with magnitude q, q' and q'', respectively located at $\vec{r} = x_0\hat{x},\ y_0\hat{y},\ z_0\hat{z}$, respectively? Recall: $\left(\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)\right)$

(a) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x-x_0}{|\vec{r}-x_0\hat{x}|^3} + q'\frac{x+x_0}{|\vec{r}+x_0\hat{x}|^3} + q''\frac{x}{|\vec{r}|^3}\right]$,

(b) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x-x_0}{|\vec{r}-x_0\hat{x}|^3} + q'\frac{y-y_0}{|\vec{r}-y_0\hat{y}|^3} + q''\frac{z-z_0}{|\vec{r}-z_0\hat{z}|^3}\right]$,

(c) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x-x_0}{|\vec{r}-x_0\hat{x}|^3} + q'\frac{x}{|\vec{r}-y_0\hat{y}|^3} + q''\frac{x}{|\vec{r}-z_0\hat{z}|^3}\right]$

(d) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x}{|\vec{r}-x_0\hat{x}|^3} + q'\frac{y}{|\vec{r}-y_0\hat{y}|^3} + q''\frac{z}{|\vec{r}-z_0\hat{z}|^3} \right]$

(e) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x}{|\vec{r}-x_0\hat{x}|^3} + q'\frac{x}{|\vec{r}-y_0\hat{y}|^3} + q''\frac{x}{|\vec{r}-z_0\hat{z}|^3} \right]$

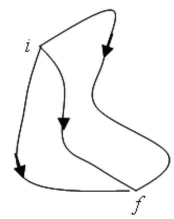
4 Electric Potential

4.1 Conservative force and potential energy

Recall that in Newtonian mechanics, a potential energy can be defined for a particle that obeys Newton's Law $\vec{F} = ma$ if the force is *conservative*. In this chapter, we shall see that the electric force felt by charges (we assume implicitly that charges have mass and obey Newton's Law) are conservative. Therefore, we can define potential energy for charges moving under electric force. We can also define the *electric potential – the potential energy per unit charge*, following the introduction of electric field from electric force. First we review what a conservative force is.

Conservative force

A conservative force is a force where the work done by the force in moving a particle (that experience the force) is path independent. It depends only on the initial and final positions of the particle.



In this case, the work done by the force on the particle can be expressed as minus the difference between the *potential energies* on the two end points, $-\Delta W = U_f - U_i$, the potential energy $U(\vec{x})$ is a function of the position of the particle only. The potential energy can be related to the force by noting that the work done is related to the force by $dW = \vec{F} \bullet d\vec{l}$. Consequently,

$$W = \int_i^f \vec{F} \bullet d\vec{l} = -(U_f - U_i) \qquad \text{(E.12).}$$

The integral is path independent for conservative force. Using multi-variable calculus, it can be shown that the above equation can be inverted to obtain $\vec{F} = -\nabla U(\vec{r})$. This is a simplified notation for 3 equations

$$F_x = -\frac{\partial U(x,y,z)}{\partial x}, \quad F_y = -\frac{\partial U(x,y,z)}{\partial y}, \quad F_z = -\frac{\partial U(x,y,z)}{\partial z} \qquad \text{(E.12a).}$$

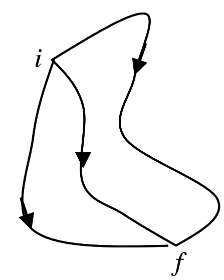
4.2 Electric Potential energy and electric potential

We can define electric potential energy for a point charge particle if electric force is conservative. Fortunately we know that electric force is conservative because of its similarity to gravitational force, $\vec{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_o r_{12}^2}\hat{r}_{12}$. We know from analogy with gravitational force that the electric potential energy for a charge q_1 moving under the influence of another charge q_2 should be $U_{12} = \frac{q_1 q_2}{4\pi\varepsilon_o r_{12}}$, and the corresponding electric potential at a distance r from a charge q_2 is $V_2(r) = \frac{U_{12}}{q_2} = \frac{q_2}{4\pi\varepsilon_o r}$.

Direct Evaluation of V_2.

The potential energy expression U_{12} can be derived using the formula

$$W = \int_i^f \vec{F} \bullet d\vec{s} = -(U_f - U_i),$$



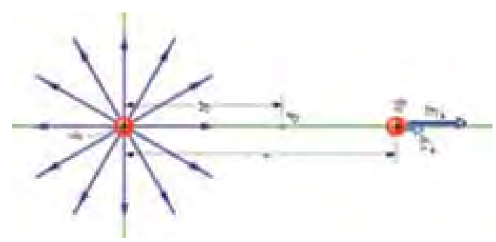
where we shall set the initial point i at infinity and the final point f at a distance r from a fixed charge q_2. Notice that the electric potential is given by a similar integral

$$V_f - V_i = -\int_i^f \vec{E} \bullet d\vec{s},$$

since electric field is the force per unit charge by definition. We shall for simplicity chooses the initial and final points to be on a straight line that extends radially from q_2 (see figure).

In general, the work done by the electric field W on a charge q moving from point $\vec{r}_1$ where the electric potential is $V(\vec{r}_1)$ to point $\vec{r}_2$ with potential $V(\vec{r}_2)$ is $W = qV(\vec{r}_1) - qV(\vec{r}_2)$.

In this case the integral becomes

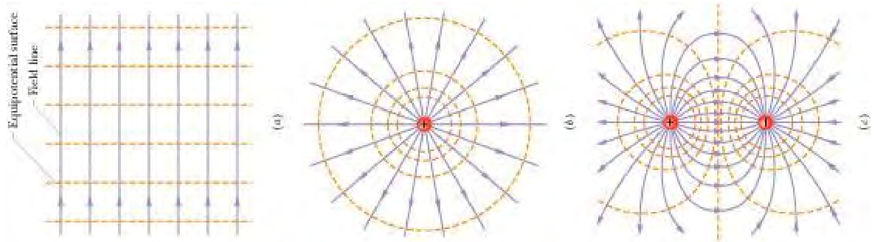


$$V_f - V_i = -\frac{q_2}{4\pi\varepsilon_o}\int_\infty^r \frac{1}{r^2} dr = \frac{q_2}{4\pi\varepsilon_o r}.$$

Choosing the reference electric potential to be $V_i = V(r = \infty) = 0$, we obtain the result $V_2(r) = \frac{q_2}{4\pi\varepsilon_o r}$ (E.13).

Equipotential Surfaces

The construction of equipotential surfaces is useful in this case. Equipotential surfaces are points in space that have the same electric potential. These points usually form closed surfaces in space. For the electric potential from a single charge q, $V(r) = \frac{q}{4\pi\varepsilon_o r}$. The equipotential surfaces are surfaces consist of points at same distance r from the charge, i.e. they are *spherical surfaces* surrounding the charge.



Notice that no (net) work *W* is done on a charged particle when the particle moves between any two points *i* and *f* on the same equipotential surface. This follows from

$$W = \int_i^f \vec{F}.d\vec{s} = -(U_f - U_i) = 0 \text{ if } U_f = U_i.$$

As a result the electric force (field) must be everywhere *perpendicular* to the equipotential surface. To see this we consider an arbitrary infinitesimal displacement $d\vec{s}$ on the equipotential surface. The work done is $dW = \vec{F} \bullet d\vec{s} = 0$. This can be satisfied for arbitrary displacement $d\vec{s}$ on the equipotential surface only if the force $\vec{F}$ is everywhere perpendicular to the equipotential surface.

The figure above shows the equipotential surfaces for several different situations. The surfaces can be constructed easily from the electric field lines since they are surfaces perpendicular to the electric field (force) lines.

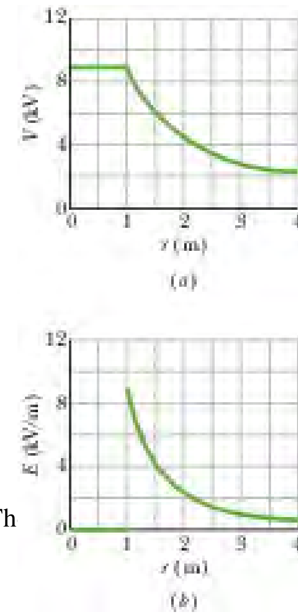
Potential of a charged Isolated Conductor

Recall that for excess charges distributed on an isolated conductor, all charges must be distributed on the surface such that the electric field on the surface cannot have a component parallel to the surface. (Otherwise there will be a current on the conductor surface)

Therefore, for any points *i* and *f* on the surface of conductor,

$$V_f - V_i = -\int_i^f \vec{E} \bullet d\vec{s} = 0,$$

i.e. the surface of an isolated conductor forms a equipotential surface. We show in the figure below the electric potential *V(r)* and electric field *E(r)* as a function of distance *r* from center for a spherical conductor with radius 1.



Notice that $E(r) \sim \dfrac{\partial V(r)}{\partial r}$.

Units of electric potential energy and electric potential:

The SI unit for energy is Joule (J) (1 Joule = 1 Newton × 1 meter). Th tial is therefore

1 *Volt (V) = 1 Joule per Coulomb (J/C).*

This new unit allows us to adopt a more conventional unit for electric field $\vec{E}$, which we have measured up to now in Newton per Coulomb (N/C). We note that

$$1 \quad N/C = \left(1\frac{N}{C}\right)\left(\frac{1V.C}{1J}\right)\left(\frac{1J}{1N.m}\right) = 1 \quad V/m.$$

We also notice that an energy unit that is often more convenient to use in atomic and subatomic domain is electron-volt (eV). 1 eV = energy equal to the work required to move a single elementary charge e through a potential difference of one volt, or

1 eV = e ×(1V) = 1.6×10^{-19}*C)(1J/C)=*1.6×10^{-19}*J.*

4.3 Electric Potential due to a group of point charges

For many charges, the forces and electric field satisfy the law of superposition,

$$\vec{E}(\vec{r}) = \sum_j E_j(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j).$$

Therefore, the electric potential $V(\vec{r}) - V_\infty = -\int_\infty^{\vec{r}} \vec{E} \bullet d\vec{s}$ is given by (setting $V_\infty = 0$)

$$V(\vec{r}) = -\frac{1}{4\pi\varepsilon_o}\sum_j q_j \int_{\infty}^{\vec{r}} \frac{\vec{r}\,'-\vec{r}_j}{|\vec{r}\,'-\vec{r}_j|^3} \bullet d\vec{s}\,' = \frac{1}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|} \qquad \text{(E.14).}$$

The last equality comes because the final potential is a sum of potentials from single point charges. Notice that electric potential is a *scalar* and the last sum is a simple algebraic sum of numbers (scalars). This is considerably simpler to evaluate than the sum over electric fields which are *vectors*.

Example: electric potential at position $\vec{r} = (x, y, z)$ from 2 point charges with magnitude q, and q', respectively located at $\vec{r} = \pm z_0 \hat{z}$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|} = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{|\vec{r}-z_0\hat{z}|} + \frac{q'}{|\vec{r}+z_0\hat{z}|}\right]$$

where $|\vec{r} \pm z_0\hat{z}| = \sqrt{x^2 + y^2 + (z \pm z_0)^2}$

Evaluating the electric field from V(r)

Recall that for a conservative force, the force can be derived from the potential energy using

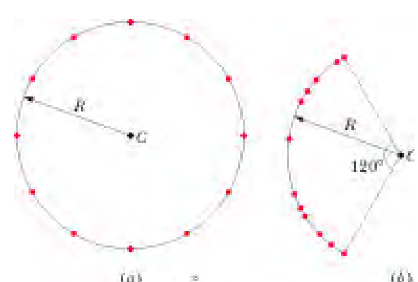
$$F_x = -\frac{\partial U(x,y,z)}{\partial x}, \quad F_y = -\frac{\partial U(x,y,z)}{\partial y}, \quad F_z = -\frac{\partial U(x,y,z)}{\partial z}.$$

Since $q\vec{E} = \vec{F}$ and $qV = U$, we also have

$$E_x = -\frac{\partial V(x,y,z)}{\partial x}, \quad E_y = -\frac{\partial V(x,y,z)}{\partial y}, \quad E_z = -\frac{\partial V(x,y,z)}{\partial z}.$$

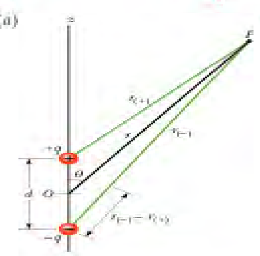
Example:

What are the electric potential and electric field at the center of the circle due to electrons in the figure below?



(Notice that there are 12 electrons around the perimeter in both cases)
Exercise: evaluate the electric field for the 2-charge problem from *V(r)*. Show that it gives the same result as what we have obtained previously by directly adding up the electric fields from 2 separate charges.

Electric dipole potential



This is the electric potential due to 2 point charges with magnitude q, and $-q$, respectively located at $\vec{r} = \pm \frac{d}{2}\hat{z}$, and at distances $|\vec{r}| >> d$ from the origin.

Using the above result, we obtain for the dipole electric potential;

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{|\vec{r}-\frac{d}{2}\hat{z}|}-\frac{1}{|\vec{r}+\frac{d}{2}\hat{z}|}\right] \underset{r>>d}{\sim} \frac{qd(\vec{r}.\hat{z})}{4\pi\varepsilon_o r^3} = \frac{p\cos\vartheta}{4\pi\varepsilon_o r^2}$$

where $p = qd$, and θ is the angle between $\vec{r}$ and z-axis (see figure).

Notice that in Cartesian coordinate,

$$V(\vec{r}) \underset{r>>d}{\sim} \frac{qd(\vec{r}.\hat{z})}{4\pi\varepsilon_o r^3} = \frac{pz}{4\pi\varepsilon_o (x^2+y^2+z^2)^{3/2}} \qquad \text{(E.15).}$$

4.5 Electric potential energy of a system of point charges

We have shown that for a pair of charges q_1, q_2, the electric potential energy is $U_{12} = \frac{q_1 q_2}{4\pi\varepsilon_o r_{12}}$, which is minus the work done by the electrostatic force to move one of the charges from infinity (with the other charge fixed) to its position at r_{12}. For a collection of charges, we may define the electric potential energy similarly. We define:

The electric potential of a system of fixed point charges is equal to the work done by an external agent to assemble the system, bring each charge in from an infinite distance.

Let's see how we can derive the potential energy expression for 3 charges, q_1, q_2, q_3, located at r_1, r_2, r_3, respectively, using the expression for U_{12} and the principle of superposition.

Our strategy is to bring in the charges one by one, and sum up the energy we needed to bring in all the charges together.

First we bring in charge 1 to position r_1. The electric potential energy U_1 we needed is zero, since there is no charge around.

Next we bring in the second charge to position r_2. The electric potential energy we need is $U_2=U_{12}$ from definition.

We then bring in the third charge. From the principle of superposition, the force experienced by the third charge is the sum of the forces from the first two charges. Consequently the work done needed to bring in the third charge is equal to sum of the work done against the force of the first 2 charges, i.e. $U_3=U_{13}+U_{23}$.

Therefore the total electric potential energy needed to build up the system is

$U = U_1+U_2+U_3 = (0)+(U_{12})+(U_{13}+U_{23}) = U_{12}+U_{13}+U_{23}$.

In fact, we can continue this construction to show that the electric potential of a system of N charges is

$$U = \sum_{i<j} U_{ij} = \sum_{i<j}\frac{q_i q_j}{4\pi\varepsilon_o r_{ij}} = \frac{1}{2}\sum_{i\neq j}\frac{q_i q_j}{4\pi\varepsilon_o r_{ij}}.$$

Question

The figure below shows the electric potential V as a function of x. (a) Rank the five regions according to the magnitude of the x component of the electric field within them, greatest first. What is the direction of

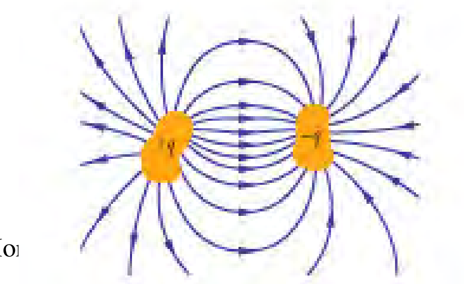
the field along the x-axis in (b) region 2 and (c) region 4? (d) Indicate points where charges may be present in the system.

5 Capacitor

5.1 Introduction

In the previous few chapters we have learnt about the basic physics in electrostatics. In this and the next few chapters we shall discuss how these physics can be applied in daily life to build circuits. We start with learning a basic circuit-device element - capacitor. Crudely speaking, capacitors are devices for storing electric charges. Because of electrostatic forces energy is also stored in capacitor at the same time.

Capacitance



Generally speaking, capacitors are composed of two isolated conductors of any shape. We shall call them capacitor plates. The capacitor is "charged" when equal and opposite amount of charges is put on the two plates.

Because the plates are conductors, they are equipotential surfaces. Mo between the two plates when the capacitor is charged.

The potential difference between the two plates can be expressed as

$$V = -\int_{+}^{-} \vec{E} \bullet d\vec{s} \ ,$$

where the integral is performed along a line joining the plate with positive charge to the plate with negative charge. Since the electric field from a charge distribution is directly proportional to the magnitude of charge, i.e., $\vec{E} \propto q$, we must have $V \propto q$, or

$$q = CV \qquad \text{(E.16).}$$

The proportionality constant C is called the capacitance of the capacitor. We shall see that C depends only on the *geometry of the plates* and not on their charge or potential difference. This is a direct consequence of the linearity of the Coulomb's Law.

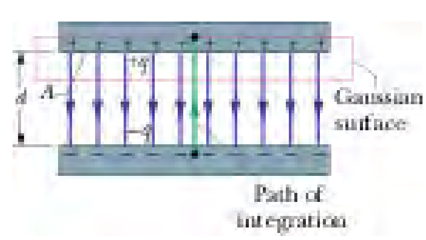
The SI unit for capacitance is *farad*.

1 farad = 1 F = 1 Coulomb per volt = 1C/V.

5.2 Calculating the Capacitance

The idea is to apply formula $V = -\int_{+}^{-} \vec{E} \bullet d\vec{s} = \int_{=}^{+} \vec{E} \bullet d\vec{s}$, for a given charge q on the capacitor, where + and – are end points of the conductor. The capacitance is deduced from the relation between V and q. We shall consider situations with high symmetry and will often apply Gauss' Law to calculate the electric field.

A Parallel-Plate capacitor



We assume that the plates of our parallel-plate capacitor are so large and so close together that we can "forget" that the plates has a boundary, i.e. we treat them as two infinite parallel plates put close together.

We draw a closed surface as shown in figure, Notice that one side of the surface is inside the conductor where $\vec{E} = 0$. In this case, all electric field come out perpendicular to the surface. Applied Gauss' Law, we have

$q = \varepsilon_o EA$, A = area of capacitor plate.

We also have

$$V = \int_{-}^{+} \vec{E} \bullet d\vec{s} = Ed \qquad \text{(E.17a)},$$

where d = distance between two plates. Therefore, $q = \varepsilon_o \left(\frac{V}{d} \right) A = CV$. The capacitance is

$$C = \frac{\varepsilon_o A}{d} \qquad \text{(E.17b)}.$$

Notice that the capacitance does depend only on geometrical factors, the plate area A and plate separate d. We shall see that this remains true in later examples.

Isolated conductors

We can define a capacitance to a single isolated conductor by taking ("removing") one piece of the conductor to infinity. For an isolated spheres taking $b \to \infty$ we obtain

$$C = 4\pi\varepsilon_o a \qquad \text{(E.18)}.$$

To understand the meaning of this result we look at the equation $q/C = V$. In the previous cases V is the potential difference between the two pieces of conductors. What is the meaning V when $b \to \infty$?

(ans: V is the potential difference between the charged conductor and *infinity*.

5.3 Capacitors in parallel and in series

When there is a combination of capacitors in a circuit, we can sometimes describe its behavior as an *equivalent capacitor* – a single capacitor that has the same capacitance as the actual combination of capacitors. In this way, we can simplify the circuit and make it easier to analyze. There are two basic configurations of capacitors that allow such a replacement.

Capacitors in parallel

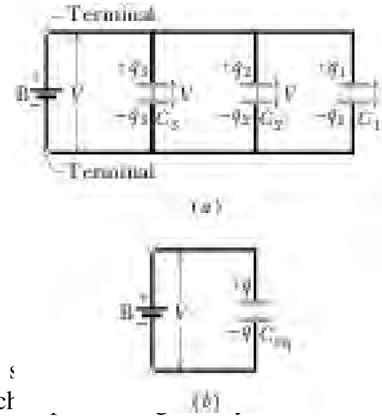
The configuration is shown in the figure.

The equivalent capacitance can be derived by analyzing the applied voltage and charge on each capacitor.

The applied voltage is the same for the (3) capacitors, i.e., we have

$q_1 = C_1 V$, $q_2 = C_2 V$ $q_3 = C_3 V$,.... (E.19a)

The total charge stored is $q = q_1 + q_2 + q_3 + \ldots\ldots$. Therefore the effective capacitance is



$$C_{eq} = \frac{q}{V} = C_1 + C_2 + C_3 + \ldots \qquad \text{(E.19b)}$$

for capacitors in parallel.

Capacitors in Series

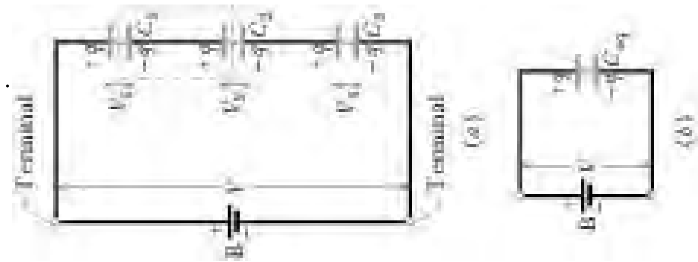
The configuration is shown in the figure below.

Notice that because of overall charge neutrality, the charge *q* s he same = *charge stored in the equivalence capacitor*. The voltage across eacl

$$V_1 = q/C_1,\ \ V_2 = q/C_2 \ \ \ V_3 = q/C_3, \ldots.$$

The total voltage across the equivalence capacitor is

$$V = V_1 + V_2 + V_3 + \ldots = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ..\right) = \frac{q}{C_{eq}} .$$



Therefore the equivalent capacitance is given by

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \qquad \text{(E.20)}$$

5.4 **Energy Stored in Capacitors**

To charge up a capacitor, we need work done. You may imagine that we need to move the electrons (-*Ve* charges) from neutral atoms in one capacitor plate to another. Energy is needed in this process because we have to separate positive and negative charges from an originally charge neutral configuration. In practice the work done is supplied by a battery.

The total energy needed to charge up a capacitor can be evaluated by noting that the energy needed to move unit charge dq against a potential V is, by definition

$$dW = Vdq = \frac{q}{C}dq .$$

Therefore the work required to bring the total capacitor charge up to final value Q is

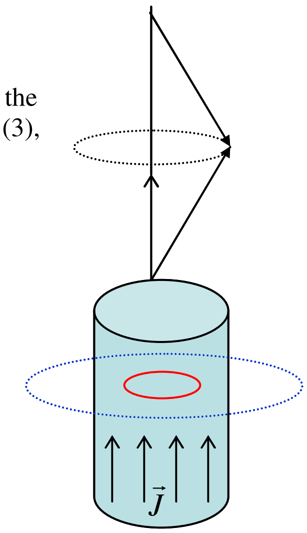
$$W = \int dW = \frac{1}{C}\int_0^Q qdq = \frac{Q^2}{2C} \qquad \text{(E.21)}$$

= potential energy (U) stored in a capacitor with charge Q.

Alternatively, using the relation $Q = CV$, we may also write $U = \frac{1}{2}CV^2$. (E.21a)

Magnetic Field

1 The B-field of an infinitely long wire



Symmetry analysis shows that the B-field must be horizontal and circles around the wire, and depends on *r* (the distance of the position to the wire) only. Using Eq. (3),

$$\oint \vec{B}\cdot d\vec{l} = 2\pi r B = \mu_0 \iint_S \vec{J}\cdot d\vec{S} = \mu_0 I, \text{ so } B = \frac{\mu_0 I}{2\pi r} \qquad \text{(M.1)}.$$

Example-1A
Find the B-field of an infinitely long cylindrical conductor of radius *R* carrying uniform current density *J*.

Solution:
Take a loop of radius *r*, as shown (the red or blue circle).
For $r > R$, the answer is the same as above with $I = J\pi R^2$.
For $r < R$, the current within the loop is $J\pi r^2$. So

$$B = \frac{\mu_0 J r}{2}. \qquad \text{(M.2)}$$

2 Force of B-field on current wire and charged particle (Lorentz force)

Lorentz force on a point charge *q* is

$$\vec{F} = q\vec{v}\times\vec{B} \qquad \text{(M.3)},$$

where $\vec{v}$ is the velocity of the charge.

2.1 Motion of a charged particle in uniform magnetic field

(a) Circular motion

When $\vec{v}$ of a particle with mass *m* and charge *q* is perpendicular to $\vec{B}$, the force is also perpendicular to $\vec{v}$, and the charge is moving in a circle. The radius of the circle *R* is determined by

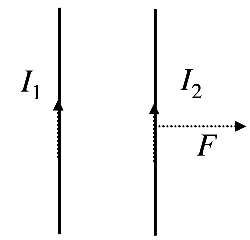
$$qvB = m\frac{v^2}{R}, \text{ or } R = mv/qB. \text{ The time period is } T = \frac{2\pi R}{v} = \frac{2\pi m}{qB} \qquad \text{(M.4)}$$

(b) General case $\vec{v}$ and $\vec{B}$ at an angle θ

Break $\vec{v}$ into a component parallel to $\vec{B}$ and a component perpendicular to $\vec{B}$. For example, choose $\vec{B}$ along the z-direction, then $\vec{v} = v\cos\theta\,\vec{z}_0 + \vec{v}_{xy}$. The force is in the x-y direction so along z-axis the particle is moving at constant speed $v\cos\theta$. Change to the reference frame S' that is moving at $v\cos\theta$ along z-axis, we then have a particle with velocity $\vec{v}_{xy}$. Using the results in (a) the motion is then a spiral with radius $R = \dfrac{mv\sin\theta}{qB}$

and distance between two spirals is $Tv\cos\theta = \dfrac{2\pi mv\cos\theta}{qB}$. Note that the period T is velocity independent.

2.2 On a thin wire in uniform B-field, it is $\vec{F} = I(\vec{l} \times \vec{B})$



Force per unit length between two parallel wires carrying currents I_1 and I_2, and separated by a distance D.

Solution:

From Example 1A, the magnetic field due to I_1 at I_2 is $B = \dfrac{\mu_0 I_1}{2\pi D}$, and pointing into the paper plane. The force is then pointing to the right, and is the same along the wire. For any section of the wire of length L, the force is $F = \dfrac{\mu_0 I_1 I_2}{2\pi D} L$ (M.5).

3 Electromagnetic Induction

3.1 Lorentz force on moving charge

For a section of electric current I in a thin wire $d\vec{l}$ is $Id\vec{l}$, the force is

$$d\vec{F} = Id\vec{l} \times \vec{B} \qquad \text{(M.6)}$$

Electromotive force $\vec{f}_s$ – any force on a charged particle other than that due to other charges.

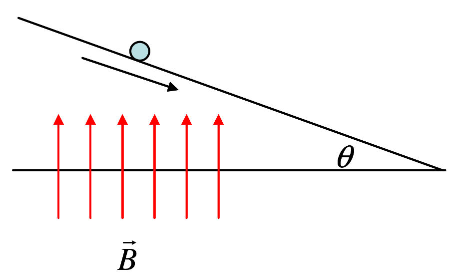
Electromotance $\xi = \int_a^b \vec{f}_s \cdot d\vec{l}$ (M.7).

Conventionally called emf, but it is not really a force. The Chinese translation 電動勢, is more appropriate.

The emf in a section of wire due to B-field is $\xi = \int_a^b (\vec{v} \times \vec{B}) \cdot d\vec{l}$ (M.8)

Example-1

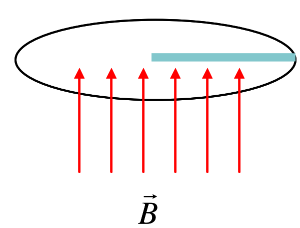
A rod of length L is sliding down a slope in a uniform B-field at speed v. Find emf in the rod.



Solution:

The amplitude of $(\vec{v} \times \vec{B})$ is $vB\cos\theta$ and its direction is pointing straight out of the paper plane, parallel to the length of the rod. So emf = $LvB\cos\theta$

Example-1A



The same rod is spinning at angular speed ω in the B-field around one of its end. The angular momentum is parallel to the B-field. Find emf between the two ends of the rod.

Solution:

Take a small length of the rod dr at distance r from the fixed end. The speed of it is ωr so its emf is $B\omega rdr$. Total emf along the whole length is $\xi = \int_0^L B\omega rdr = \frac{1}{2}B\omega L^2$ ans.

1.2 Faraday's law – A changing magnetic field induces an electric field.

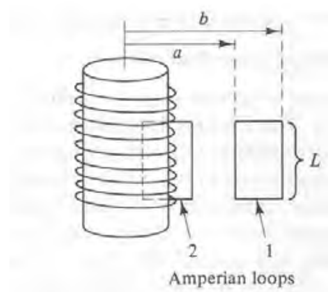
Their relation is given by:

$$\xi = \oint_l \vec{E} \cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = -\frac{d}{dt}\iint_S \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt} \qquad \text{(M.9)},$$

where 'S' is the surface enclosed by the closed line '*l*'.

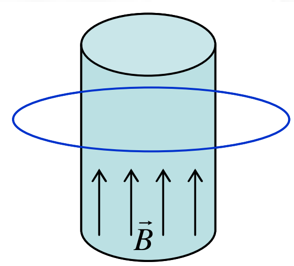
It can be shown that when a wire loop is moving relative to the B-field, Eqs. (3) and (4) are equivalent. However, Eq. (4) is more general, and works even when there is no relative motion, or the wire loop can be at the location where there is no B-field.

Example-2
The current in a long solenoid (N turns per unit length) is decreasing linearly with time t, $I = I_0 - kt$. Find the induced E-field.



Solution

By symmetry argument, we can see that the B-field is along the axis of the solenoid. Take an Ampere's loop-1 outside the solenoid, we find that the B-field is constant. But far away the B-field must be zero, so the B-field outside is zero. Now take loop-2, $BL = \mu_0 NIL$, so $B = \mu_0 N(I_0 - kt)$



Note that outside the solenoid there is no B-field, but the changing B-field induces an E-field so that if the circular wire carries charge, it will spin. Note also that Eq. (4) is of the same in mathematical form as Ampere's law, when $-\frac{\partial \vec{B}}{\partial t}$ is viewed as 'electric current' and the induced E-field as the 'magnetic field'. Applying Eq. (4), the left hand side = $2\pi rE$, where r is the radius of the loop, and the right hand side = $\mu_0 NkA$, where A is the cross section area of the solenoid. So $E = \frac{\mu_0}{2\pi}\frac{kNA}{r}$. Ans.

1.3 Self and mutual inductance (of coils)

Consider a coil (a loop of wire) carrying current I. If I changes with time, its magnetic field, and therefore the magnetic flux Φ through the coil (See Eq. (4)) will change, inducing an emf. As the field (hence flux) is proportional to I, we can define a quantity L which depends only on the geometrics of the coil, called self-inductance, such that

$$\Phi = LI \qquad \text{(M.10)},$$

because $\xi = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$ (M.10a).

The mutual inductance between two coils, M_{12} and M_{21}, are similarly defined.

$\Phi_2 = M_{12}I_1$, and $\Phi_1 = M_{21}I_2$. (Φ_2 is the magnetic flux through coil-2 due to the current I_1 in coil-1).
(M.11)

It can be shown that $M_{12} = M_{21}$, and they depend only on the coils geometrics. (M.12)

Example-4
The magnetic flux through a coil of resistance R changes uniformly from Φ_1 to Φ_2 over a time interval T, Find the total charge passing through any cross section of the wire.

Solution:

From Eq. (6) and (6A) $-\frac{d\Phi}{dt} = \frac{\Phi_1 - \Phi_2}{T} = \xi = RI = R$, so $Q = IT = (\Phi_1 - \Phi_2)/R$. ans.