香港資優教育學苑
The Hong Kong Academy for Gifted Education

International Mathematical Olympiad Preliminary Selection Contest - Hong Kong 2012

2012 國際數學奧林匹克 – 香港選拔賽

26 May 2012 (Saturday)
2012 年 5 月 26 日（星期六）

Question Book
問題簿

Instructions to Contestants:
考生須知：

1. The contest comprises a 3 hours written test.
比賽以筆試形式進行，限時三小時。

2. Questions are in bilingual versions. Contestants should answer all questions.
題目中英對照。參賽學生必須解答全卷所有題目。

3. Put your answers on the answer sheet.
請將答案寫在答題紙上。

4. The use of calculators is NOT allowed.
不可使用計算機。

5. Measuring instruments like rulers, compasses, etc. can be used.
直尺、圓規及其它量度工具可作輔助之用。

Co-organised by The Hong Kong Academy for Gifted Education,
the Gifted Education Section of the Education Bureau and
International Mathematical Olympiad Hong Kong Committee
香港資優教育學苑、教育局資優教育組及國際數學奧林匹克香港委員會合辦

1. Let n be a two-digit number such that the square of the sum of digits of n is equal to the sum of digits of n^2. Find the sum of all possible values of n. (1 mark)

設 n 為兩位數，而 n 的數字之和的平方等於 n^2 的數字之和。求 n 所有可能值之和。（1 分）

2. A positive integer n is said to be 'good' if n^2-1 can be written as the product of three distinct prime numbers. Find the sum of the five smallest 'good' integers. (1 mark)

對於正整數 n，若 n^2-1 可寫成三個不同質數之積，則 n 稱為「好數」。求最小的五個「好數」之和。（1 分）

3. There are 13 students in a class (one of them being the monitor) and 13 seats in the classroom. Every day, the 13 students line up in random order and then enter the classroom one by one. Except for the monitor, each student will randomly choose an unoccupied seat and sit down. The monitor, however, prefers the seat next to the door and chooses it if possible. What is the probability that the monitor can choose his favourite seat? (1 mark)

某班有 13 名學生（其中一人為班長），課室內有 13 個座位。每天上課時，這 13 名學生均按隨意的次序排隊進入課室。除班長外，每人進入課室後均會隨意選一個空位坐下，唯班長特別喜歡近門口的一個座位，若未有人選擇的話他便會選那個座位。班長能選到他喜歡的座位的概率是多少？（1 分）

4. For positive integer n, let $D(n)$ be the eventual value obtained when the digits of n (in base 10) are added up recursively until a one-digit number is obtained. For example $D(4)=4$, $D(2012)=D(5)=5$ and $D(1997)=D(26)=D(8)=8$. If x denotes the 2012th Fibonacci number (i.e. the 2012th term of the sequence 1, 1, 2, 3, 5, 8, 13, …), find $D(x)$. (1 mark)

對於正整數 n，設 $D(n)$ 表示不斷迭代計算 n 的數字之和直至得到一個一位數的最終結果，例如：$D(4)=4$、$D(2012)=D(5)=5$、$D(1997)=D(26)=D(8)=8$。若 x 表示第 2012 個斐波那契数（即數列 1, 1, 2, 3, 5, 8, 13, … 的第 2012 項），求 $D(x)$。（1 分）

5. In a school there are 300 boys and 300 girls, divided into 5 classes, each with the same number of students. It is known that there are at least 33 boys and 33 girls in each class. A boy and a girl from the same class may form a group to enter a contest, and each student may only belong to one group. What is the maximum number of groups that can be guaranteed to form? (1 mark)

某校有男生和女生各 300 名，他們被分成 5 班，每班人數相同。已知每班均最少有男生和女生各 33 名。同班的一名男生和一名女生可組隊參加一項比賽，而每名學生只可隸屬一隊。保證能夠組成的隊伍數目的最大值是多少？（1 分）

6. If $x^3-3\sqrt{2}x^2+6x-2\sqrt{2}-8=0$, find the value of x^5-41x^2+2012. (1 mark)

若 $x^3-3\sqrt{2}x^2+6x-2\sqrt{2}-8=0$，求 x^5-41x^2+2012 的值。（1 分）

7. Find the smallest positive integer which cannot be expressed in the form $\frac{2^a - 2^b}{2^c - 2^d}$ where a, b, c, d are non-negative integers. (1 mark)

求最小的正整數，該數不能表達成 $\frac{2^a - 2^b}{2^c - 2^d}$ （其中 a、b、c、d 為非負整數）的形式。（1 分）

8. Let $f(x) = x^2 + 12x + 30$. Find the largest real root of the equation $f(f(f(x))) = 0$. (1 mark)

設 $f(x) = x^2 + 12x + 30$。求方程 $f(f(f(x))) = 0$ 的最大實根。（1 分）

9. How many different ways are there to rearrange the letters in the word 'BRILLIANT' so that no two adjacent letters are the same after the rearrangement? (1 mark)

有多少種不同的方法可把「BRILLIANT」一字中的字母重新排列，使得排列後沒有兩個相鄰的字母相同？（1 分）

10. If $\frac{\sin x}{\sin y} = 3$ and $\frac{\cos x}{\cos y} = \frac{1}{2}$, find $\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$. (1 mark)

若 $\frac{\sin x}{\sin y} = 3$ 及 $\frac{\cos x}{\cos y} = \frac{1}{2}$，求 $\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$。（1 分）

11. An n-digit number x has the following property: if the last digit of x is moved to the front, the result is $2x$. Find the smallest possible value of n. (2 marks)

某 n 位數 x 具以下性質：若把 x 的最後一位數字移到最前，所得的數是 $2x$。求 n 的最小可能值。（2 分）

12. In ΔABC, $AB = 3$, $BC = 4$ and $CA = 5$. A circle cuts AB at C_1 and C_2, cuts BC at A_1 and A_2, and cuts CA at B_1 and B_2. If $A_1A_2 = B_1B_2 = C_1C_2 = x$, and that the area of the hexagon formed by the vertices A_1, A_2, B_1, B_2, C_1 and C_2 is 4, find x. (2 marks)

在 ΔABC 中，$AB = 3$、$BC = 4$、$CA = 5$。某圓與 AB 交於 C_1 和 C_2，與 BC 交於 A_1 和 A_2，與 CA 交於 B_1 和 B_2。若 $A_1A_2 = B_1B_2 = C_1C_2 = x$，且由 A_1、A_2、B_1、B_2、C_1、C_2 組成的六邊形的面積為 4，求 x。（2 分）

13. $ABCD$ is a convex quadrilateral in which AC and BD meet at P. Given $PA = 1$, $PB = 2$, $PC = 6$ and $PD = 3$. Let O be the circumcentre of ΔPBC. If OA is perpendicular to AD, find the circumradius of ΔPBC. (2 marks)

$ABCD$ 是凸四邊形，其中 AC 與 BD 交於 P。已知 $PA = 1$、$PB = 2$、$PC = 6$ 及 $PD = 3$。設 O 為 ΔPBC 的外心。若 OA 與 AD 垂直，求 ΔPBC 的外接圓半徑。（2 分）

14. Given a, b, c, d (where $a<b<c<d$) are positive integers not exceeding 2012, and they form an arithmetic sequence in this order. How many different sets of possible values of (a, b, c, d) are there? (2 marks)

已知 a、b、c、d（其中 $a<b<c<d$）均為不超過 2012 的正整數，且它們按此順序組成一個等差數列。問 (a, b, c, d) 有多少組不同的可能值？ （2 分）

15. $ABCD$ is a convex quadrilateral and E, F are the mid-points of BC and CD respectively. The line segments AE, AF and EF divide $ABCD$ into four triangles, whose areas are four consecutive integers. Find the greatest possible area of ΔABD. (2 marks)

$ABCD$ 是凸四邊形，E、F 分別是 BC 和 CD 的中點。線段 AE、AF 和 EF 把 $ABCD$ 分成四個三角形，它們的面積分別是四個連續整數。求 ΔABD 的面積的最大可能值。 （2 分）

16. Let D be a point inside ΔABC such that $\angle BAD = \angle BCD$ and $\angle BDC = 90°$. If $AB = 5$, $BC = 6$ and M is the midpoint of AC, find the length of DM. (2 marks)

設 D 為 ΔABC 內的一點，使得 $\angle BAD = \angle BCD$ 及 $\angle BDC = 90°$。若 $AB = 5$，$BC = 6$，且 M 是 AC 的中點，求 DM 的長度。 （2 分）

17. In ΔABC, $\angle C = 90°$, $\angle BAC < 45°$ and $AB = 7$. P is a point on AB such that $\angle APC = 2\angle ACP$ and $CP = 1$. Find $\frac{AP}{BP}$. (2 marks)

在 ΔABC 中，$\angle C = 90°$，$\angle BAC < 45°$ 且 $AB = 7$。P 是 AB 上的一點，使得 $\angle APC = 2\angle ACP$ 及 $CP = 1$。求 $\frac{AP}{BP}$。 （2 分）

18. In how many different ways can we rearrange the twelve integers 1 to 12 on the face of a clock so that the sum of any three adjacent integers after the rearrangement is divisible by 3? (2 marks)

有多少種不同的方法可把鐘面上 1 至 12 等 12 個整數重新排列，使得排列後任意三個相鄰的整數之和皆可被 3 整除？ （2 分）

19. Find a factor of 100140001 which lies between 8000 and 9000. (2 marks)

求 100140001 一個介乎 8000 和 9000 之間的因數。 （2 分）

20. How many different triangles with integer side lengths are there such that the sum of the lengths of any two sides exceeds the length of the third side by at least 5 units, and that the area is numerically twice the perimeter? (Two triangles are regarded to be the same if they are congruent.) (2 marks)

有多少個不同的三角形各邊的長度均為整數，任何兩邊的長度之和均比第三邊長 5 單位或以上，且其面積在數值上是其周界的兩倍？（兩個全等的三角形視為相同。） （2 分）

– End of Paper – 全卷完 –